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Quadratization of symmetric pseudo-Boolean functions

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A B S T R A C T

A *pseudo-Boolean function* is a real-valued function $f(x) = f(x_1, x_2, \ldots, x_n)$ of n binary variables, that is, a mapping from $\{0, 1\}^n$ to $\mathbb{R}$. For a pseudo-Boolean function $f(x)$ on $\{0, 1\}^n$, we say that $g(x, y)$ is a *quadratization* of f if $g(x, y)$ is a quadratic polynomial depending on x and on m *auxiliary* binary variables $y_1, y_2, \ldots, y_m$ such that $f(x) = \min\{g(x, y) : y \in \{0, 1\}^m\}$ for all $x \in \{0, 1\}^n$. By means of quadratizations, minimization of f is reduced to minimization (over its extended set of variables) of the quadratic function $g(x, y)$. This is of practical interest because minimization of quadratic functions has been thoroughly studied for the last few decades, and much progress has been made in solving such problems exactly or heuristically. A related paper by Anthony et al. (2015) initiated a systematic study of the minimum number of auxiliary y-variables required in a quadratization of an arbitrary function f (a natural question, since the complexity of minimizing the quadratic function $g(x, y)$ depends, among other factors, on the number of binary variables). In this paper, we determine more precisely the number of auxiliary variables required by quadratizations of *symmetric* pseudo-Boolean functions $f(x)$, those functions whose value depends only on the Hamming weight of the input x (the number of variables equal to 1).

1. Introduction

1.1. Quadratizations of pseudo-Boolean functions

A *pseudo-Boolean function* is a real-valued function $f(x) = f(x_1, x_2, \ldots, x_n)$ of n binary variables, that is, a mapping from $\{0, 1\}^n$ to $\mathbb{R}$. It is well-known that every pseudo-Boolean function can be uniquely represented as a multilinear polynomial in its variables. *Nonlinear binary optimization problems*, or *pseudo-Boolean optimization* (PBO) *problems*, of the form

$$\min\left\{f(x) : x \in \{0, 1\}^n\right\},$$

where $f(x)$ is a pseudo-Boolean function, have attracted the attention of numerous researchers, and they are notoriously difficult, as they naturally encompass a broad variety of models such as maximum satisfiability, maximum cut, graph coloring, simple plant location, and so on. Many approaches have been proposed for the solution of PBO problems; the reader may refer to [3,5,17] for general overviews. In recent years, several authors have revisited an approach initially proposed by Rosenberg [20]. This involves reducing PBO to its quadratic case (QPBO) by relying on the following concept.

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Definition 1.1. For a pseudo-Boolean function $f(x)$ on $\{0,1\}^n$, we say that $g(x,y)$ is a *quadratization* of f if $g(x,y)$ is a quadratic polynomial depending on x and on m *auxiliary* binary variables $y_1, y_2, \ldots, y_m$, such that

$$f(x) = \min\left\{g(x,y) : y \in \{0,1\}^m\right\} \quad \text{for all } x \in \{0,1\}^n.$$

Clearly, if $g(x,y)$ is a quadratization of f, then

$$\min\left\{f(x) : x \in \{0,1\}^n\right\} = \min\left\{g(x,y) : x \in \{0,1\}^n, y \in \{0,1\}^m\right\},$$

so that the minimization of f is reduced through this transformation to the QPBO problem of minimizing $g(x,y)$ (much like linearization techniques reduce QPBO to a linear optimization problem in 0–1 variables; see [3,5]).

We are also interested (see [1]) in special types of quadratizations, which we call *y-linear quadratizations*, which contain no products of auxiliary variables. If $g(x,y)$ is a y-linear quadratization, then g can be written as

$$g(x,y) = q(x) + \sum_{i=1}^{m} a_i(x) y_i,$$

where $q(x)$ is quadratic in x and each $a_i(x)$ is a linear function of x. When minimizing g over y, each product $a_i(x)y_i$ takes the value $\min\{0, a_i(x)\}$.

Rosenberg [20] has proved that every pseudo-Boolean function $f(x)$ has a quadratization, and that a quadratization can be efficiently computed from the polynomial expression of f. This also easily follows from our observations in Section 1.2 that every monomial has a quadratization. (It is also the case – see [1] – that any pseudo-Boolean function has a y-linear quadratization.) Of course, quadratic PBO problems remain difficult in general, but this special class of problems has been thoroughly studied for the last few decades, and much progress has been made in solving large instances of QPBO, either exactly or heuristically. Quadratization has emerged in recent years as one of the most successful approaches to the solution of very large-scale PBO problems arising in computer vision applications. (See, for instance, Boykov, Veksler and Zabih [4], Kolmogorov and Rother [15], Kolmogorov and Zabih [16], Rother, Kolmogorov, Lempitsky and Szummer [22], Kohli, Ladický and Torr [13], Kohli, Pawan Kumar and Torr [14], Boros and Gruber [2], Fix, Gruber, Boros and Zabih [7], Freedman and Drineas [8], Ishikawa [10,11], Ramalingam, Russell, Ladický and Torr [19], Rother, Kohli, Feng and Jia [21], Kappes et al. [12].)

In a related paper, the present authors [1] initiated a systematic study of quadratizations of pseudo-Boolean functions. We investigated the minimum number of auxiliary y-variables required in a quadratization. We showed, in particular, that there are pseudo-Boolean functions of n variables for which every quadratization must involve at least $\Omega(2^{n/2})$ auxiliary variables and, conversely, that $O(2^{n/2})$ auxiliary variables always suffice for every function. Other authors have established more precise upper bounds for special subclasses of pseudo-Boolean functions: $(n-1)$ auxiliary variables for symmetric functions (Fix [6]), $\lfloor \frac{n-1}{2} \rfloor$ auxiliary variables for positive monomials (Ishikawa [11]), two auxiliary variables for certain types of submodular functions (Kohli et al. [13], Ramalingam et al. [19]), etc. In this paper, our focus is on *symmetric* pseudo-Boolean functions. We introduce this class in the next section.

1.2. Symmetric functions

A symmetric pseudo-Boolean function is one in which the value of the function depends only on the weight of the input. More precisely, a pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ is *symmetric* if there is a discrete function $k : \{0,1,\ldots,n\} \to \mathbb{R}$ such that $f(x) = k(l)$ where $l = |x| = \sum_{j=1}^{n} x_j$ is the Hamming weight (number of ones) of x. In another way, f is symmetric if it is invariant under any permutation of the coordinates $\{1,2,\ldots,n\}$ of its variables. Here, we investigate the number of auxiliary variables required in quadratizations of symmetric functions.

Example 1.2. Consider the *negative monomial*

$$-\prod_{i=1}^{n} x_i = -x_1 x_2 \cdots x_n.$$

This elementary symmetric pseudo-Boolean function has the following *standard quadratization* which requires only one auxiliary variable (Freedman and Drineas [8]):

$$s_n(x_1, x_2, \ldots, x_n, y) = y\left(n - 1 - \sum_{i=1}^{n} x_i\right).$$

The reason is as follows: unless all the x_i are 1, then the quantity in parentheses in the expression for s_n is non-negative and the minimum value of s_n is therefore 0, obtained when $y = 0$; and, if all x_i are 1, the expression equals $-y$, minimized when $y = 1$, giving value -1. In both cases, the minimum value of s_n is the same as the value of the negative monomial. □