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Regular independent sets

Yair Caro [a], Adriana Hansberg [b,*], Ryan Pepper [c]

[a] Department of Mathematics and Physics, University of Haifa-Oranim, Tivon 36006, Israel
[b] Instituto de Matemáticas, UNAM Juriquilla, 76230 Querétaro, Mexico
[c] University of Houston-Downtown, Houston, TX 77002, United States

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ABSTRACT

The regular independence number, introduced by Albertson and Boutin in 1990, is the size of a largest set of independent vertices with the same degree. Lower bounds were proven for this invariant, in terms of the order, for trees and planar graphs. In this article, we generalize and extend these results to find lower bounds for the regular k-independence number for trees, forests, planar graphs, k-trees and k-degenerate graphs.

1. Introduction and benchmark bounds

Albertson and Boutin [2] introduced the parameter $\alpha_{\text{reg}}(G)$ as the cardinality of a largest independent set of vertices with equal degree in a graph G. An independent set whose vertices all have equal degree in G is called a *regular independent set*. A *k-independent set* is a set of vertices whose induced subgraph has maximum degree at most k. Let us define the *regular k-independence number*, denoted by $\alpha_{k-reg}(G)$, as the maximum cardinality of a k-independent set of vertices which have the same degree in G. More particularly, we denote by $\alpha_{k,j}(G)$ the maximum cardinality of a k-independent set in the subgraph induced by the vertices of degree j in G. Thus, $\alpha_{k-reg}(G) = \max\{\alpha_{k,j}(G) : \delta \le j \le \Delta\}$, where δ is the minimum degree and Δ is the maximum degree. When $k = 0$, $\alpha_{0-reg}(G) = \alpha_{reg}(G)$ and, for regular graphs, $\alpha_{\text{reg}}(G) = \alpha(G)$ and $\alpha_{k-\text{reg}}(G) = \alpha_k(G)$. Albertson and Boutin [2] proved the following:

(1) If G is a planar graph on n vertices, then $\alpha_{\text{reg}}(G) \ge \frac{2}{65}n$.
(2) If G is a maximal planar graph on n vertices, then $\alpha_{\text{reg}}(G) \ge \frac{3}{61}n$.
(3) If T is a tree on n vertices, then $\alpha_{\text{reg}}(T) \ge \frac{n+2}{4}$ and this is sharp.
(4) If G is a connected graph on n vertices and maximum degree Δ, then $\alpha_{\text{reg}}(G) \ge \frac{n}{\binom{\Delta+1}{2}}$ and this is sharp.

Albertson and Boutin left as open problems whether the results in (1) and (2) are best possible and constructed an example of a maximal planar graph G for which $\alpha_{\text{reg}}(G) = \frac{n}{16}$. Our intention is to extend the issue raised by Albertson and Boutin in two directions. We extend the question to broader families of graphs, including forests, k-trees and k-degenerate graphs, while on the way improving item (1) for minimum degree $\delta = 2, 3, 4, 5$ and item (2) for minimum degree $\delta = 4, 5$.

* Corresponding author.
E-mail addresses: yacaro@kvgeva.org.il (Y. Caro), ahansberg@im.unam.mx (A. Hansberg), pepperr@uhd.edu (R. Pepper).

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These results are summarized in Table 2 in what follows. We also extend the problem from regular independent sets to regular k-independent sets, in view of some recent results about the k-independence number [10,22]. We mention in passing that related papers were written, from distinct inspiration (see for example [1,4,7,8,12]), from which some lower bounds on the regular k-independence number can be obtained, but which are inferior to the lower bound obtained by the current approach. Furthermore, the relevance of our results to Complexity Theory is demonstrated in Section 5, where we prove that the decision problems of determining if a graph G has a k-independent set or a regular k-independent set of size at least p are both NP-complete in general. We also show that, for a given hereditary family of graphs, if the first problem is polynomial time solvable in this family, then so is the second problem. As a secondary motivation for our research, one could consider marketing analysis of social networks. In such cases, there may be much interest on profiles of groups of users showing similarity in some indicators — like activity and number of "friend-followers" in the network, which is given by their degree in the associated "friendship-graph". Locating independence in such groups may help in seeding viral messages.

The parameter $\alpha_{\rm reg}(G)$ is closely related to a newly introduced parameter [11], the fair domination number $\mathrm{fd}(G)$. A *fair dominating set* is a set $S \subseteq V(G)$ such that all vertices $v \in V(G) \setminus S$ have exactly the same non-zero number of neighbors in S. The *fair domination number* $\mathrm{fd}(G)$ is the cardinality of a minimum fair dominating set of G. When $\delta(G) \geq 1$ and R is a maximum regular independent set of G, then $V(G) \setminus R$ is a fair dominating set of G and hence $\mathrm{fd}(G) \leq n - \alpha_{\rm reg}(G)$. Hence, any lower bound on $\alpha_{\rm reg}(G)$ yields an upper bound on $\mathrm{fd}(G)$ (and any lower bound on $\mathrm{fd}(G)$ yields an upper bound on $\alpha_{\rm reg}(G)$).

A few more definitions are necessary before proceeding. The *repetition number* of G, denoted as $\mathrm{rep}(G)$, was introduced in [12] and is defined as the maximum number of vertices with equal degree, while $\chi_k(G)$ is the *k-chromatic number of* G, i.e. the minimum number of colors needed to color the vertices of the graph G such that the graphs induced by the vertices of each color class have maximum degree at most k. Note that $\chi_0(G)$ is the well-known chromatic number $\chi(G)$.

The following proposition serves as our starting point for this paper. It gives a rather general lower bound, and giving improvements to this lower bound is a major task of this paper.

Proposition 1.1. *Let G be a graph with average degree d and minimum degree δ. Then*

$$\alpha_{k-\rm reg}(G) \geq \frac{n}{(2d - 2\delta + 1)\chi_k(G)}.$$

Proof. Let G_i be the subgraph of G induced by all vertices of degree i, $\delta \leq i \leq \Delta$. Then we have

$$\alpha_{k-\rm reg}(G) = \max\{\alpha_{k,j}(G) \;:\; \delta \leq j \leq \Delta\} \geq \max\left\{\frac{|V(G_j)|}{\chi_k(G_j)} \;:\; \delta \leq j \leq \Delta\right\},$$

since, for every j, $\alpha_{k,j}(G) \geq \frac{|V(G_j)|}{\chi_k(G_j)}$ holds. Hence, with

$$\max\left\{\frac{|V(G_j)|}{\chi_k(G_j)} \;:\; \delta \leq j \leq \Delta\right\} \geq \max\left\{\frac{\mathrm{rep}(G)}{\chi_k(G_j)} \;:\; |V(G_j)| = \mathrm{rep}(G),\; \delta \leq j \leq \Delta\right\} \geq \frac{\mathrm{rep}(G)}{\chi_k(G)}$$

and the bound $\mathrm{rep}(G) \geq \frac{n}{2d-2\delta+1}$ proved in [12], the result follows. $\square$

A comparison of the benchmark results obtained in Proposition 1.1 to the outcomes of our more intricate approach is presented in subsequent Tables 2 and 3.

Definitions for terms not defined in the introduction but necessary for the paper appear in the sections in which they are needed. The remainder of this paper is organized as follows. In Section 2 we deal with trees and forests, where we generalize and extend the results of Albertson and Boutin in [2] to $\alpha_{k-\rm reg}(G)$. In Section 3, we present a lower bound on $\alpha_{\rm reg}(G)$ for k-trees, and give analogous results for k-degenerate graphs and planar graphs, by refining and elaborating previous methods to generalize and improve the bounds given in [2]. In Section 4 we give lower bounds on $\alpha_{2-\rm reg}(G)$ for planar and outerplanar graphs by incorporating ideas from defective colorings [15,16] and a partition theorem due to Lovász [23]. In Section 5, we analyze complexity issues of regular k-independence and we finish with a collection of open problems on regular k-independent sets in Section 6.

2. Trees and forests

In this section, we generalize and extend the result that $\alpha_{reg}(T) \geq \frac{n+2}{4}$ for any tree T, obtained by Albertson and Boutin in [2], to regular k-independence number in both trees and forests. Here and throughout, we will use $n_i(G)$ to denote the number of vertices of degree i in G. When the context is clear, $n_i(G)$ will be abbreviated to n_i. As usual, a *leaf* of a tree T is a vertex of degree 1 in T and its neighbor is called its *support vertex*.

Theorem 2.1. *For every tree T on $n \geq 2$ vertices,*

(i) $\alpha_{\rm reg}(T) \geq \frac{n+2}{4}$ *(Albertson and Boutin [2]),*
(ii) $\alpha_{1-\rm reg}(T) \geq \frac{2(n+2)}{7}$, *and*
(iii) $\alpha_{k-\rm reg}(T) \geq \frac{n+2}{3}$ *for $k \geq 2$.*

Moreover, all bounds are sharp.