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Richardson's Theorem for k-colored kernels in strongly connected digraphs

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ABSTRACT

A digraph $D = (V, A)$ is said to be *m-colored* if its arcs are colored with m colors. An m-colored digraph D has a *k-colored kernel* if there exists $K \subseteq V$ such that (i) for every $v \in V \setminus K$ there exist a q-colored directed path, with $q \leq k$, from v to a vertex of K, and (ii) for every pair $\{u, v\} \subseteq K$ every directed path from u to v uses at least $k + 1$ colors.

Given an m-colored digraph D, the *color-class digraph* of D, denoted $\mathbf{C}(D)$, is defined as follows: the vertices of $\mathbf{C}(D)$ are the m colors of D, and (i, j) is an arc of $\mathbf{C}(D)$ if and only if there exist two consecutive arcs in D, namely (u, v) and (v, w), such that (u, v) has color i and (v, w) has color j.

A digraph D is said to be *cyclically k-partite* if there is a partition $\{V_i\}_{i=0}^{k-1}$ of its vertices in independent sets such that every arc in D is either a loop or a V_iV_{i+1}-arc (taken the index modulo k). In Galeana-Sánchez (2012) it was proved that given an m-colored digraph D, if $\mathbf{C}(D)$ is cyclically 2-partite then D has a kernel by monochromatic paths (that is a 1-colored kernel). In this paper we extend this work and prove the following: *Let D be a strongly connected m-colored digraph D such that, for some $k \geq 1$, $\mathbf{C}(D)$ is a cyclically $(k+1)$-partite digraph, with partition $\{\mathbf{C}_i\}_{i=0}^{k}$.* (i) *If for some part $\mathbf{C}_j$, no vertex of $\mathbf{C}_j$ has a loop, then D has a k-colored kernel.* (ii) *For each i, with $0 \leq i \leq k$, let D_i be the subgraph of D induced by the set of arcs with color in $\mathbf{C}_i$, and for each vertex x of D let $N^+_{\mathbf{C}}(x)$ and $N^-_{\mathbf{C}}(x)$ be the set of colors appearing in the ex-arcs and in-arcs of x, respectively. If for some subgraph D_j, for every vertex x of D_j we have that $N^+_{\mathbf{C}}(x) \not\subseteq N^-_{\mathbf{C}}(x)$, then D has a k-colored kernel.*

As a direct consequence we obtain a proof of Richardson's Theorem in the case D is strongly connected, and a proof of a classical result by M. Kwaśnik (see Kwaśnik (1983)) on the existence of k-kernels (a k-kernel of a digraph $D = (V, A)$ is a set $S \subseteq V$ such that for any $v \in V \setminus S$, $d_D(v, S) \leq k - 1$ and for every pair $\{u, v\} \subseteq S$, $d_D(u, v) \geq k$) that asserts that if D is a strongly connected digraph such that every directed cycle has length congruent with 0 modulo k, then D has a k-kernel.

1. Introduction

A digraph $D = (V, A)$ is an *m-colored digraph* if its arcs are colored with m-colors. A non empty set $S \subseteq V$ is a *k-colored absorbing set* if for every vertex $u \in V \setminus S$ there exists $v \in S$ such that there is a q-colored directed path from u to v, with $q \leq k$.

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A non empty set $S \subseteq V$ is called a *k-colored independent set* if for every pair $\{u, v\} \subseteq S$, there is no q-colored directed path from u to v, with $q \leq k$. A set $K \subseteq V$ is called *k-colored kernel* if K is a k-colored absorbing set and a k-colored independent set.

This notion was introduced in [7] and it is a natural generalization to kernels by monochromatic paths (the case of 1-colored kernels) defined first in [14]. It is also a generalization of the classical notion of kernels in digraphs which has been extensively studied (see [3,5] for a recent remarkable survey on the topic). Moreover, this concept also generalizes the concept of *k-kernel*. A set $S \subseteq V$ is a k-kernel of D if it is k-independent (that means that for any pair $\{u, v\} \subseteq S$, $d_D(u, v) \geq k$) and $(k-1)$-absorbing (that means for any $u \in V \setminus S$ there exists $v \in S$ such that $d_D(u, v) \leq k-1$). The concept of k-kernel was introduced by Borowiecki and Kwaśnik in [10] and in [11] Kwaśnik proved the now classical result that asserts that if D is a strongly connected digraph such that every directed cycle has length congruent with 0 modulo k then D has a k-kernel. The study of 1-colored kernels in m-colored digraphs already has relatively extensive literature. Also the k-kernels in digraphs were widely studied (see for example [4,8,9,12,15,16]).

In [6] was introduced the concept of *color-class digraph* of an m-colored digraph D, denoted $\mathbf{C}(D)$, as follows: the vertices of $\mathbf{C}(D)$ are the colors represented in the arcs of D and (i, j) is an arc of $\mathbf{C}(D)$ if and only if there exist two consecutive arcs in D, namely (u, v) and (v, w), such that (u, v) has color i and (v, w) has color j. This concept was introduced mainly to prove that if $\mathbf{C}(D)$ is a bipartite digraph then D has a kernel by monochromatic paths. Later, in [9] was proved the following result: If D is a m-colored digraph such that $\mathbf{C}(D)$ has no direct cycles of odd length at least 3 then D has a kernel by monochromatic paths (a 1-colored kernel).

In this paper we study structural properties of the digraph $\mathbf{C}(D)$ that guarantee the existence of a k-colored kernel and we prove the following results.

Theorem 1.1. *Let $D = (V, A)$ be an strongly connected m-colored digraph, and suppose that, for some $k \geq 1$, its color-class digraph $\mathbf{C}(D)$ is a cyclically $(k+1)$-partite digraph, with partition $\{\mathbf{C}_i\}_{i=0}^{k}$. If for some j, with $0 \leq j \leq k$, no vertex of $\mathbf{C}_j$ has a loop, then D has a k-colored kernel.*

Theorem 1.2. *Let $D = (V, A)$ be an strongly connected m-colored digraph, and suppose that, for some $k \geq 1$, its color-class digraph $\mathbf{C}(D)$ is a cyclically $(k+1)$-partite digraph, with partition $\{\mathbf{C}_i\}_{i=0}^{k}$. For each i, with $0 \leq i \leq k$, let $D_i = (V_i, A_i)$ be the minimal subdigraph of D induced by the set of arcs with color in $\mathbf{C}_i$. If for some j, with $0 \leq j \leq k$, for every $x \in V_j$ we have that $N^+_{\mathbf{C}}(x) \not\subseteq N^-_{\mathbf{C}}(x)$, then D has a k-colored kernel.*

These results generalize Richardson's Theorem on kernels in strongly connected digraphs and also generalizes the result of Kwaśnik on the existence of k-kernels in digraphs mentioned above. Also it is proved that all the hypotheses are thight for the cases when $k \geq 2$.

2. Notation

Let $D = (V, A)$ be a digraph. A directed path $P = (x_0, x_1, \ldots, x_n)$ of D will be called an *x_0x_n-path*. Given $S_1, S_2 \subseteq V$, an arc xy of D with $x \in S_1$ and $y \in S_2$ will be called an *S_1S_2-arc*, and an *S_1S_2-path* is an xy-path where $x \in S_1$ and $y \in S_2$ (if $S_1 = \{x\}$ we will write xS-path and Sx-path instead of $\{x\}S$-path and $S\{x\}$-path, respectively). Given $x \in V$, let $F^-(x) = \{yx : yx \in A\}$; $F^+(x) = \{xy : xy \in A\}$ and $F(x) = F^+(x) \cup F^-(x)$.

We will said that the digraph $D = (V, A)$ is *m-colored* if its arcs are colored with m colors. A non empty set $S \subseteq V$ is a *k-colored absorbing set* if for every vertex $u \in V \setminus S$ there is a uS-path with at most k colors. A non empty set $S \subseteq V$ is called a *k-colored independent set* if for every pair $u, v \in S$, every uv-directed path uses at least $k+1$ colors. A set $K \subseteq V$ is called *k-colored kernel* if K is a k-colored absorbing set and a k-colored independent set.

A digraph $D = (V, A)$ will be called *cyclically k-partite* if there is a partition $\{V_i\}_{i=0}^{k-1}$ of V in independent sets such that every arc in D is either a loop or a V_iV_{i+1}-arc (taken the index modulo k).

Given an arc-coloring $\mathbf{C} : A \to \{1, \ldots, m\}$ of D, for each $x \in V$ let $N^+_{\mathbf{C}}(x) = \{\mathbf{C}(e) : e \in F^+(x)\}$ and $N^-_{\mathbf{C}}(x) = \{\mathbf{C}(e) : e \in F^-(x)\}$.

For general concepts we refer the reader to [1,2].

3. Preliminary results

Let $D = (V, A)$ be a strongly connected m-colored digraph and suppose that its color-class digraph $\mathbf{C}(D) = (V_{\mathbf{C}}, A_{\mathbf{C}})$ is a cyclically $(k+1)$-partite digraph with partition $\{\mathbf{C}_i\}_{i=0}^{k}$, and for each i, with $0 \leq i \leq k$, let $D_i = (V_i, A_i)$ be the minimal subdigraph of D such that $A_i = \{e \in A : \mathbf{C}(e) \in \mathbf{C}_i\}$. Observe that by definition, for every i, with $0 \leq i \leq k$, there is no isolated vertex in D_i and, for every $x \in V$, $x \in V_i$ if and only if $F(x) \cap A_i \neq \emptyset$ if and only if $N_{\mathbf{C}}(x) \cap \mathbf{C}_i \neq \emptyset$. Moreover, since no pair of vertices in $\mathbf{C}_i$ are adjacent, it follows that every path in D_i is monochromatic.

Lemma 3.1. *Let $x \in V$.*

(a) *Either $N^+_{\mathbf{C}}(x) \cap N^-_{\mathbf{C}}(x) = \emptyset$ or $N^+_{\mathbf{C}}(x) \subseteq N^-_{\mathbf{C}}(x)$ or $N^-_{\mathbf{C}}(x) \subseteq N^+_{\mathbf{C}}(x)$.*

(b) *There is i, with $0 \leq i \leq k$, such that $F(x) \subseteq A_i$ if and only if $N^-_{\mathbf{C}}(x) = N^+_{\mathbf{C}}(x)$ and $|N^-_{\mathbf{C}}(x)| = 1$.*