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Counting the number of non-equivalent vertex colorings of a graph

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ABSTRACT

We give some extremal properties on the number $\mathcal{B}(G)$ of non-equivalent ways of coloring a given graph G, also known as the (graphical) Bell number of G. In particular, we study bounds on $\mathcal{B}(G)$ for graphs with a maximum degree constraint. First, an upper bound on $\mathcal{B}(G)$ is given for graphs with fixed order n and maximum degree Δ. Then, we give lower bounds on $\mathcal{B}(G)$ for fixed order n and maximum degree $1, 2, n-2$ and $n-1$. In each case, the bound is tight and we describe all graphs that reach the bound with equality.

1. Introduction

A question which probably sounds familiar for many researchers in graph theory is: what is the number of ways of coloring a given graph G? In the literature, the common answer to this question is related to the notion of chromatic polynomial which was introduced by Birkhoff [1] in 1912 in an attempt to prove the four-color theorem. The *chromatic polynomial* of a graph G of order n is the (unique) polynomial of degree n whose graph passes through the points $(k, \Pi(G, k))$ for $k = 0, 1, \ldots, n$, where $\Pi(G, k)$ is the number of ways of coloring G with at most k colors, counting two colorings as distinct when they are obtained by a permutation from the other. For example, for the path P_3 we have

$$\Pi(\mathrm{P}_3, k) = k(k-1)^2.$$

Indeed, $\Pi(\mathrm{P}_3, 0) = \Pi(\mathrm{P}_3, 1) = 0$; $\Pi(\mathrm{P}_3, 2) = 2$ (take for instance the first two colorings in the left column of Fig. 1) and $\Pi(\mathrm{P}_3, 3) = 12$ as shown in Fig. 1. The number of vertex colorings of a graph G is nowadays commonly interpreted as $\Pi(G, n)$, where n is the number of vertices in G, meaning that P_3 has 12 colorings according to this interpretation. However, for P_3, another answer to the above question that makes sense is *two* as depicted in Fig. 2: there is only one coloring with two colors (the two extremities share the same color while the central vertex has its own color), and only one coloring with three colors (each vertex has its own color).

More generally, we are interested in this paper by the number $\mathcal{B}(G)$ of *non-equivalent* colorings of a graph G. This way of counting colorings is especially meaningful when a set of elements has to be partitioned into a given number of non-empty subsets, subject to some constraints. Indeed, $\mathcal{B}(G)$ is the number of partitions of the vertex set of G whose blocks are stable sets. This invariant has been studied by several authors in the last few years [7–10] under the name of *(graphical) Bell number* of G. However, historically, this invariant is related to the σ-polynomial introduced by Korfhage [11] in 1978. Indeed, the σ-*polynomial* of a graph G is a polynomial in x such that the coefficient of x^k is the number of non-equivalent ways of properly

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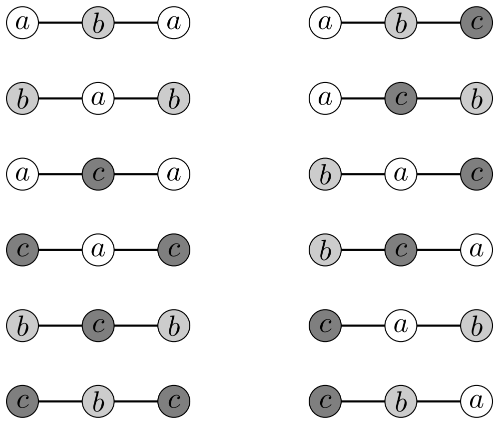

Fig. 1. The 12 colorings of P_3 (using 3 colors) as defined by the chromatic polynomial.

Fig. 2. The 2 non-equivalent colorings of P_3 (using any number of colors).

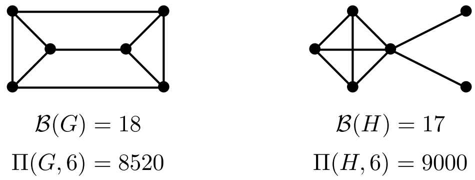

Fig. 3. Two graphs G and H with 6 vertices such that $\Pi(G, 6) < \Pi(H, 6)$ and $\mathcal{B}(G) > \mathcal{B}(H)$.

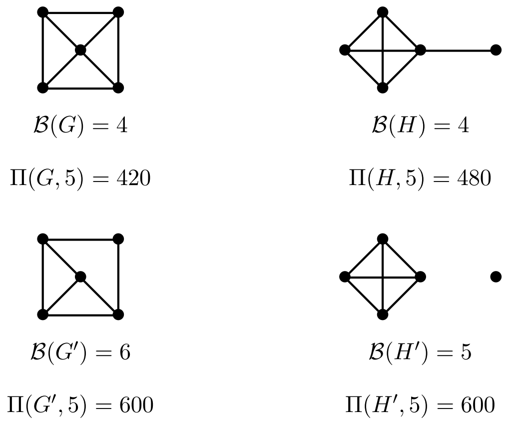

Fig. 4. Two pairs of graphs with 5 vertices showing that equality for one way to count the colorings does not imply equality for the other.

coloring G using exactly k colors. It follows from that definition that $\mathcal{B}(G)$ is the value of the σ-polynomial at $x = 1$. The σ-polynomial was studied intensively by Brenti [3] and Brenti et al. [4] in the early nineties. It appears that several results on $\mathcal{B}(G)$ published later (including results in [7–9]) are special cases of results from [3,4].

It is interesting to note that while $\mathcal{B}(G)$ and $\Pi(G, n)$ might appear as similar concepts (since they both count colorings with at most n colors), they differ in several ways. We have already mentioned that only non-equivalent colorings are counted in $\mathcal{B}(G)$, which means that $\mathcal{B}(G)$ corresponds to the number of partitions of the vertex set of G, taking into account constraints that prevent some pairs of vertices of belonging to the same subset of the partition. Also, observe that if $\Pi(G, n) < \Pi(H, n)$ for two graphs G and H of order n, this does not necessarily imply that $\mathcal{B}(G) < \mathcal{B}(H)$ (and conversely) as shown in Fig. 3.

Similarly, there exist pairs of graphs (G, H) such that $\mathcal{B}(G) = \mathcal{B}(H)$ but $\Pi(G, n) \neq \Pi(H, n)$, and conversely (see examples in Fig. 4).