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On the independence polynomial of the corona of graphs[☆]

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ABSTRACT

Let $\alpha(G)$ be the cardinality of a largest independent set in graph G. If s_k is the number of independent sets of size k in G, then $I(G;x) = s_0 + s_1x + \cdots + s_\alpha x^\alpha$, $\alpha = \alpha(G)$, is the *independence polynomial* of G (Gutman and Harary, 1983). $I(G;x)$ is *palindromic* if $s_{\alpha-i} = s_i$ for each $i \in \{0, 1, \ldots, \lfloor \alpha/2 \rfloor\}$. The *corona* of G and H is the graph $G \circ H$ obtained by joining each vertex of G to all the vertices of a copy of H (Frucht and Harary, 1970).

In this paper, we show that $I(G \circ H;x)$ is palindromic for every graph G if and only if $H = K_r - e, r \geq 2$. In addition, we connect realrootness of $I(G \circ H;x)$ with the same property of both $I(G;x)$ and $I(H;x)$.

1. Introduction

Throughout this paper G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and the *size* of G are $|V(G)|$ and $|E(G)|$, respectively, while $\overline{G}$ denotes its *complement*. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X.

By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$. We also denote by $G - F$ the subgraph of G obtained by deleting the edges of F, for $F \subseteq E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$.

The *neighborhood* of a vertex $v \in V$ is the set

$$N_G(v) = \{w : w \in V \text{ and } vw \in E\}, \quad \text{and} \quad N_G[v] = N_G(v) \cup \{v\}.$$

If there is no ambiguity on G, we use $N(v)$ and $N[v]$, respectively.

K_n, P_n, C_n denote respectively, the complete graph on $n \geq 0$ vertices, the path on $n \geq 2$ vertices, and the cycle on $n \geq 3$ vertices.

The *disjoint union* of the graphs G_1, G_2 is the graph $G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1), V(G_2)$, and as edge set the disjoint union of $E(G_1), E(G_2)$. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G. The *Zykov sum* of the disjoint graphs G_1 and G_2 is the graph $G_1 + G_2$ with $V(G_1) \cup V(G_2)$ as a vertex set and $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$ as an edge set [46].

An *independent* set in G is a set of pairwise non-adjacent vertices. An independent set of maximum size is a *maximum independent set* of G, and the *independence number* $\alpha(G)$ is the cardinality of a maximum independent set in G.

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Let s_k be the number of independent sets of size k in a graph G. The polynomial

$$I(G;x) = s_0 + s_1x + s_2x^2 + \cdots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of G [19]. For a survey on independence polynomials of graphs see [25]. Some basic procedures to compute the independence polynomial of a graph are recalled in the following.

Theorem 1.1 ([19]). (i) $I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x)$;
(ii) $I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$;
(iii) $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$ *holds for every* $v \in V(G)$.

The *corona* of the graphs G and H is the graph $G \circ H$ obtained from G and $|V(G)|$ copies of H, such that each vertex of G is joined to all vertices of a copy of H [13]. More general compound graphs are considered in [40].

Theorem 1.2 ([16]). $I(G \circ H; x) = (I(H; x))^n I\left(G; \frac{x}{I(H;x)}\right)$, *where* $n = |V(G)|$.

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be:

- *unimodal* if there exists an index $k \in \{0, 1, \ldots, n\}$, called the *mode* of the sequence, such that

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n;$$

- *f-palindromic* if $a_{n-i} = f(i) \cdot a_i$ for all $i \in \{0, \ldots, \lfloor n/2 \rfloor\}$ [31];
- *palindromic* (or *self-reciprocal*) if $a_i = a_{n-i}$, $i \in \{0, \ldots, \lfloor n/2 \rfloor\}$, i.e., $f(i) = 1$ for all $i \in \{0, \ldots, \lfloor n/2 \rfloor\}$.

A polynomial is called unimodal (palindromic, f-palindromic) if the sequence of its coefficients is unimodal (palindromic, f-palindromic, respectively). For instance, the independence polynomial:

- $I(K_{127} + 3K_7; x) = 1 + 148x + \mathbf{147}x^2 + 343x^3$ is non-unimodal and non-palindromic;
- $I(K_{43} + 3K_7; x) = \mathbf{1} + 64x + 147x^2 + \mathbf{343}x^3$ is unimodal and non-palindromic;
- $I(K_{18} + 3K_3 + 4K_1; x) = 1 + \mathbf{31}x + 33x^2 + \mathbf{31}x^3 + x^4$ is unimodal and palindromic;
- $I(K_{52} + 3K_4 + 4K_1; x) = 1 + 68x + \mathbf{54}x^2 + 68x^3 + x^4$ is non-unimodal and palindromic;
- $I(P_3 \circ (K_2 \cup K_1); x) = 1 + \mathbf{12}x + 52x^2 + 105x^3 + 104x^4 + \mathbf{48}x^5 + 8x^6$ is f-palindromic for $f(i) = 2^{3-i}$, $0 \leq i \leq 3$, and unimodal.

For other examples of independence polynomials, see [1,22–24,26,30,33,43,45]. Alavi, Malde, Schwenk and Erdös proved that for every permutation π of $\{1, 2, \ldots, \alpha\}$ there is a graph G with $\alpha = \alpha(G)$ such that $s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(\alpha)}$ [1].

Conjecture 1.3 ([1]). *$I(T; x)$ is unimodal for every tree T.*

In [27] we conjectured that $I(G; x)$ is unimodal for every bipartite graph G. It turned out to be true for almost all equibipartite graphs [14]. On the other hand, a counterexample for this conjecture has been found in [5].

The palindromicity of matching polynomial and characteristic polynomial of a graph were examined in [21], while for independence polynomial we quote [18,28,32,41].

It is known that the product of two unimodal polynomials is not necessarily unimodal even for independence polynomials. For example, the graph $G = K_{95} + 3K_7$ has $I(G; x) = 1 + 116x + 147x^2 + 343x^3$, which is unimodal, while

$$I(2G; x) = 1 + 232x + 13\,750x^2 + 34\,790x^3 + 101\,185x^4 + \mathbf{100\,842}x^5 + 117\,649x^6$$

is not unimodal.

Theorem 1.4 ([2]). *If the polynomials P and Q are both unimodal and palindromic, then $P \cdot Q$ is unimodal and palindromic.*

However, the above result cannot be generalized to the case when P is unimodal and palindromic, while Q is unimodal and non-palindromic; e.g.,

$$P = 1 + x + 3x^2 + x^3 + x^4, \quad Q = 1 + x + x^2 + x^3 + 2x^4, \quad \text{while}$$
$$P \cdot Q = 1 + 2x + 5x^2 + 6x^3 + 8x^4 + 7x^5 + 8x^6 + 3x^7 + 2x^8.$$

It is worth mentioning that one can produce graphs with palindromic independence polynomials in various ways [3,17,41].

We pay special attention to graphs of independence number two. Clearly, $\alpha(H) = 2$ is equivalent to the fact that the complement of H is triangle-free. If, in addition, H is perfect, then its complement is bipartite. Notice that if H is disconnected, then H is the disjoint union of two complete graphs. A recursive structure of the family of graphs with independence number two may be found in [44]. There are situations, where the matching structure of $\alpha = 2$ graphs is of importance. For instance, it is useful, when Hadwiger's conjecture is under consideration [9,10,39].