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Exact solutions for Latency-Bounded Target Set Selection Problem on some special families of graphs ☆

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ABSTRACT

In the t-Latency-Bounded Target Set Selection (t-LBTSS) problem, we are given a simple graph $G = (V, E)$, a certain latency bound t and a threshold function $\theta(v) = \lceil \rho d(v) \rceil$ for every vertex v of G, where $0 < \rho < 1$ is a rational number and $d(v)$ is the degree of v in V, the goal is to find a target set S with smallest cardinality such that all vertices in V are activated by S by a so called "diffusion process" within t rounds as follows: Initially, all vertices in the target set become activate. Then at each step i of the process, each vertex get activated if the number of active vertices in its neighbor after $i - 1$ exceeds its threshold.

For general graphs, the t-LBTSS problem is not only NP-hard, it is also hard to be approximated by Chen's inapproachability results (Chen, 2009). In this paper, we are interested in finding an optimal target set for some special family of graphs. A simple, tight but non-trivial inequality was presented which gives the lower bound of the total sum of degrees in a feasible target set to t-LBTSS problem, in terms of the number of edges in the graph. Necessary and sufficient conditions for equality to hold have been established, based on which we are able to construct families of infinite number of graphs for which the optimal solution to t-LBTSS problem become obvious. In particular, we gave an exact formula for the optimal solution of a kind of toroidal mesh graphs, while it seems difficult to tell what the optimal solutions are for these graphs without using the equality given in the paper.

1. Introduction

Nowadays, social networks (such as Facebook, Twitter, Myspace and so on) have attracted more attention from researchers. One typical issue considered by several authors is how to spread influence through a social network. For instance, you want to market a new product and hope more people would adopt it. Initially, by targeting a few influential people, say, giving them free samples of the product, a cascade of influence can be triggered in the network — friends will recommend the product to other friends, and many individuals will ultimately try it through "word-of-mouth" effects. The problem is how to select the "few" initial people as the target set, such that the influence can be maximized.

In this paper, we consider a variant of the above problem under the deterministic diffusion model, namely, the t-Latency-Bounded Target Set Selection (t-LBTSS) problem, which is formally described as follows:

Assume that we are given a simple graph $G = (V, E)$ representing a social network and a latency bound t. For any vertex v, we have a threshold $\lceil \rho d(v) \rceil$, where ρ is a rational number with $0 < \rho < 1$ and $d(v)$ is the degree of v. Initially, all vertices in V are inactive and a subset $S := A_0(G)$ of vertices in V is selected and become activated. Suppose A_j is the subset

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of nodes that have been activated at the jth round for $j = 0, 1, \dots, i-1$. Now we describe the rule for the activation of nodes at the ith round. Let $A_i(G)$ $(i = 1, \dots, t)$ be the subset of vertices in $V \setminus \cup_{0\le j<i} A_j(G)$ such that for every vertex v in $A_i(G)$, at least $\lceil \rho d(v) \rceil$ of its neighbors lie in $\cup_{0\le j<i} A_j(G)$. Then every vertex in $A_i(G)$ is activated at the ith round. The t-LBTSS problem asks to select a subset of vertices $S =: \min - \text{seed}(G, \rho, t)$ with the minimum cardinality such that after t rounds, all vertices in V become activated.

The spread of influence in a network usually takes a long time. So the study of target set selection with latency bound is significant in some time-critical scenario. In what follows, we shall write A_i for $A_i(G)$ if no confusion arises. Moreover, the threshold is called majority threshold if $\rho = \frac{1}{2}$, namely, $\lceil \rho d(v) \rceil = \lceil \frac{d(v)}{2} \rceil$.

1.1. Related work

The influential nodes selection problem in social networks was first considered by Domingos and Richardson [12] (note a special case of the problem for $t = 1$ and $\rho = 1/2$ was first considered by Linial et al. [19]). Kempe et al. [16,17] further formulated the problem as an optimization problem called Influence Maximization Problem for influential nodes selection. The problem is to select a set of k nodes in the network which generate the largest expected cascade based on a given probabilistic diffusion model. They showed that natural greedy algorithm achieves a performance ratio of $1 - \frac{1}{e} - \varepsilon$ for selecting a target set of size k.

Chen [4] studied the Target Set Selection Problem (TSS) without latency bounded constraints, and proved a strong inapproachability result that makes unlikely the existence of an algorithm with approximation factor better than $O(2^{\log^{1-\varepsilon}|V|})$. Ben-Zwi et al. [2] proved that TSS problem can be solved in time $n^{O(\omega)}$, but not in $n^{o(\sqrt{\omega})}$ unless all problems in SNP can be solved in sub-exponential time, where ω is the treewidth of a graph. Ackerman et al. [1] showed combinatorial lower and upper bounds on the size of the Minimum Perfect Target Set Selection Problem, this improves bounds of Chang and Lyuu [3] for majority thresholds in directed graphs. Paper [8,20] considered the tractable cases of TSS. Some exact formulas or approximate bounds for TSS on some special classes of graphs are given in [6,5,7,13,18,15,21,22]. Zhang et al. [23] studied positive influence dominating sets problem in power-law graphs.

The t-LBTSS problem was first studied by Zou et al. [24], in which they showed the NP-hardness of this problem (note a similar result was obtained by Peleg [21] earlier in 2002) and gave two heuristic algorithms in the case of $t = 1$. Dinh et al. [11] proved that any feasible solution of t-LBTSS problem is a constant factor approximation for power-law graphs. Cicalese et al. [9] gave polynomial time exact algorithms to t-LBTSS problem for graphs with bounded clique-width and pointed out that Chen's inapproximability result still holds for t-LBTSS problem.

1.2. Our contributions

In this paper, we give a simple, tight but nontrivial inequality which describes the characteristics of feasible solution of the t-LBTSS problem. Moreover, we give a necessary and sufficient condition for the equality to hold, based on which, we are able to give a method to construct families of infinite number of graphs for which the optimal solution to t-LBTSS problem becomes apparent. In particular, we give an exact formula for the optimal solution of a kind of toroidal mesh graphs, while it seems difficult to tell what the optimal solutions are for these graphs without using the equality given in the paper.

2. The inequality

Given an arbitrary graph, it is hard to imagine what the exact or even approximate optimal solution to t-LBTSS problem is. Nevertheless, we do have some way to measure it indirectly. In the following theorem, we will show that the sum of degrees of any feasible target set S is lower bounded by some constant factor of the number of edges of the graph.

To simplify the description, denote by $E(A, B)$ the set of edges between two subsets of vertices A and B in the graph. Especially, we write $E(v, B)$ if $A = \{v\}$ includes only a vertex. In addition, define $d(S) := \sum_{v\in S} d(v)$, for any subset $S \subseteq V(G)$.

Theorem 1. *For any graph G, we have the following inequality*

$$d(S) \ge \frac{2\rho^t}{(1-\rho)^{t-1} + \rho(1-\rho)^{t-2} + \cdots + \rho^{t-1} + \rho^t}|E| = \begin{cases} \dfrac{2\rho^t(1-2\rho)}{(1-\rho)^t - 2\rho^{t+1}}|E|, & \rho \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right) \\ \dfrac{2}{1+2t}|E|, & \rho = \dfrac{1}{2} \end{cases}$$

where S is any feasible target set to t-LBTSS problem with latency bound t, and |E| is the number of edges in graph G.

Proof. Denote by A_i the set of vertices that are activated by the target set S at the ith round, $1 \le i \le t$. For convenience, we define $A_0 := S$ and $B_i := \cup_{0\le j\le i} A_j$ in what follows.