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A dichotomy for the dominating set problem for classes defined by small forbidden induced subgraphs

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ABSTRACT

We completely determine the complexity status of the dominating set problem for hereditary graph classes defined by forbidden induced subgraphs with at most five vertices.

1. Introduction

A *coloring* is an arbitrary mapping of colors to vertices of some graph. A graph coloring is said to be *proper* if no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimal number of colors in proper colorings of G. The *coloring problem*, for a given graph and a number k, is to determine whether its chromatic number is at most k or not. The *vertex k-colorability problem* is to verify whether vertices of a given graph can be properly colored with at most k colors. The edge k-colorability problem is defined by analogy.

An *independent set* and a *clique* of a graph are sets of pairwise non-adjacent and adjacent vertices, respectively. The *independent set problem* is to determine whether a given graph contains an independent set with a given number of elements. The clique problem is defined by analogy.

For a graph G, a subset $V' \subseteq V(G)$ *dominates* $V'' \subseteq V(G)$ if each of the vertices of $V''\backslash V'$ has a neighbor in V'. A *dominating set* of a graph G is a subset dominating all its vertices. The size of a minimum dominating set of G is said to be the *domination number* of G denoted by $\gamma(G)$. For a graph G and a number k, the *dominating set problem* is to decide whether $\gamma(G) \leq k$ or not.

A *class* is a set of simple unlabeled graphs closed under isomorphism. A class of graphs is *hereditary* if it is closed under deletion of vertices. It is well-known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and the graphs in $\mathcal{X}$ are said to be *$\mathcal{Y}$-free*. If $\mathcal{Y} = \{G\}$, then we will write "G-free" instead of "$\{G\}$-free". If a hereditary class can be defined by a finite set of forbidden induced subgraphs, then it is said to be *finitely defined*.

The coloring problem for G-free graphs is polynomial-time solvable if G is an induced subgraph of a P_4 or a $P_3 + K_1$, and it is NP-complete in all other cases [13]. A similar result is known for the dominating set problem. Namely, the problem is polynomial-time solvable for $Free(\{G\})$ if $G = P_i + O_k$, where $i \leq 4$ and k is arbitrary, and it is NP-complete for all other choices of G [11]. The situation for the vertex k-colorability problem is not clear, even when only one induced subgraph is forbidden.

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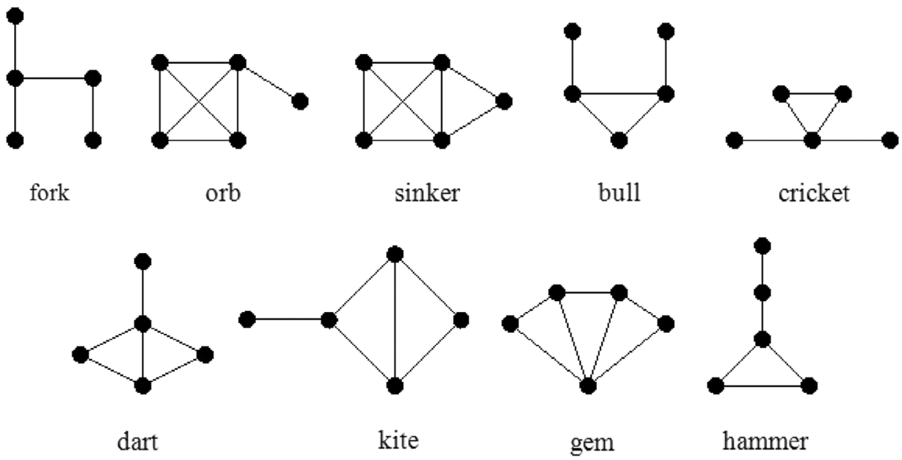

Fig. 1. The graphs *fork*, *orb* and, etc.

The complexity of the vertex 3-colorability problem is known for all the classes of the form $Free(\{G\})$ with $|V(G)| \leq 6$ [5]. A similar result for G-free graphs with $|V(G)| \leq 5$ was recently obtained for the vertex 4-colorability problem [10]. On the other hand, for fixed k, the complexity status of the vertex k-colorability problem is open for P_7-free graphs ($k = 3$), for P_6-free graphs ($k = 4$), and for $P_3 + P_2$-free graphs ($k = 5$).

The independent set problem is polynomial-time solvable for a hereditary class defined by forbidden induced subgraphs with at most five vertices if and only if a forest is one of the subgraphs, unless $P = NP$ [14,16]. A similar complete complexity dichotomy was obtained in [19] for the edge 3-colorability problem. For the coloring problem, a complete classification for pairs of forbidden induced subgraphs is open, even if they have at most four vertices. Although, the complexity is known for some such pairs [9,15,20,21,28].

In the paper, we present a complete dichotomy for the dominating set problem in the family of hereditary classes defined by forbidden induced subgraphs with at most five vertices.

2. Notation

We use the standard notation P_n, O_n, K_n for a simple path, an empty graph, and a complete graph with n vertices, respectively. A graph $K_{p,q}$ is a complete bipartite graph with p vertices in the first part and q in the second. The graphs *fork*, *orb*, *sinker*, *bull*, *cricket*, *dart*, *kite*, *gem*, *hammer* are drawn in Fig. 1.

A formula $N(x)$ denotes the neighborhood of a vertex x. A *sum* $G_1 + G_2$ is the disjoint union of G_1 and G_2 with non-intersected sets of vertices. A *graph join* $G_1 \times G_2$ of graphs with non-intersected sets of vertices is a graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{(v, u)| \ v \in V(G_1), u \in V(G_2)\})$. For a graph G and $V' \subseteq V(G)$, a graph $G[V']$ is the subgraph of G induced by V'. The symmetric difference of sets A and B is denoted by $A \otimes B$.

We refer to textbooks in graph theory for any terminology undefined here [4,7].

3. Boundary graph classes for the dominating set problem and their applications

To answer the question when an NP-complete graph problem becomes easier, a natural idea coming to mind is to consider a phase transition between easy and hard hereditary classes under some natural statements of the easiness and hardness. We use the following formal definitions. For a given NP-complete graph problem Π, a hereditary class is said to be *Π-easy* if Π can be polynomially solved for its graphs. A hereditary class is *Π-hard* if Π is NP-complete for it. Unfortunately, the phase transition approach seems to be unsuccessful.

Maximal Π-easy and minimal Π-hard classes are natural boundary elements in the lattice of hereditary classes. It turns out that the boundary may be absent at all. First, there are no maximal Π-easy classes, as any Π-easy class $\mathcal{X}$ can be extended by adding a graph $G \notin \mathcal{X}$ and all the proper induced subgraphs of G. Clearly, the resultant class is also Π-easy. Second, minimal hard classes exist for some problems and do not exist for some others. For a given graph and a positive length function on its edges, the *traveling salesman problem* is to check whether the minimum length of its cycles once visiting each vertex is at most a given number or not. It is NP-complete in the class of all complete graphs. Each proper hereditary subclass of the class is finite. Hence, the class of all complete graphs is a minimal hard case for the problem. On the other hand, for the vertex and edge variants of the k-colorability problem, any hard class contains a proper hard subclass. Indeed, if $\mathcal{Y}$ is a minimal hard case for the problem, then it must contain a graph H that cannot be properly colored in k colors. Therefore, $\mathcal{Y} \backslash Free(\{H\})$ contains only graphs that also cannot be properly colored in k colors. There is a trivial polynomial-time algorithm to test whether a given graph in $\mathcal{Y}$ belongs to $\mathcal{Y} \cap Free(\{H\})$. Hence, $\mathcal{Y} \cap Free(\{H\})$ must be hard for the