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Per-spectral and adjacency spectral characterizations of a complete graph removing six edges ☆

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ABSTRACT

Cámara and Haemers (2014) investigated when a complete graph with some edges deleted is determined by its adjacency spectrum (DAS for short). They claimed: for any $m \geq 6$ and every large enough n one can obtain graphs which are not DAS by removing m edges from a complete graph K_n. Let $\mathcal{G}_n$ denote the set of all graphs obtained from a complete graph K_n by deleting six edges. In this paper, we show that all graphs in $\mathcal{G}_n$ are uniquely determined by their permanental spectra. However, we show that for each $n \geq 7$ or $n = 5$ there is just one pair of nonisomorphic cospectral graphs in $\mathcal{G}_n$, and for $n = 4$ or 6 all graphs in $\mathcal{G}_n$ are DAS.

1. Introduction

All graphs considered in this work are undirected, finite and simple graphs. For notation and terminology not defined here, see [10].

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $A(G)$ be the *adjacency matrix* of G. The polynomial $\phi(G, x) = \det(xI - A(G))$, where I is the identity matrix, is called the *characteristic polynomial* of graph G. The *adjacency spectrum* of graph G consists of the eigenvalues of $A(G)$ together with their multiplicities. Similarly, the *permanental polynomial* of G is defined as $\pi(G, x) := \text{per}(xI - A(G))$. The *per-spectrum* of graph G, denoted by $ps(G)$, is the collection of all roots (together with their multiplicities) of $\pi(G, x)$.

Two graphs are *cospectral* (resp. *per-cospectral*) if they share the same *adjacency spectrum* (resp. per-spectrum). So two graphs are cospectral (resp. per-cospectral) if and only if they have the same characteristic (resp. permanental) polynomials. A graph G is *determined by its adjacency spectrum* (DAS for short) if every graph cospectral with G is isomorphic to G. Similarly, a graph G is said to be *determined by its per-spectrum* (DPS for short) if for any graph H, $\pi(G, x) = \pi(H, x)$ implies that H is isomorphic to G.

The characteristic polynomials of graphs and their applications are extensively examined (see for example [10]). However, only a few on the permanental polynomial and its potential applications have been published. Merris et al. [23] introduced the permanental polynomial of adjacency matrix of a general graph. The study of permanental polynomials of chemical molecular graphs were started by Kasum et al. [17]. Gutman and Cash [14] and Chen [7] obtained some relations between the coefficients of the permanental and characteristic polynomials of some chemical graphs, such as benzenoid

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hydrocarbons, fullerenes, toroidal fullerenes and coronoid hydrocarbons. For more studies on permanental polynomials, see [1,5,6,9,18,27,34,35].

The spectrum of a graph encodes useful combinatorial information about the given graph, and the relationship between the structural properties of a graph. Which graphs are determined by the spectrum? In 1956 Günthard and Primas [13] first raised the question in a paper that relates the theory of graph spectra to Hückel's theory from chemistry. van Dam and Haemers [30,31] gave an excellent survey of answers to the question of which graphs are determined by the spectra of some matrices associated to the graphs. In particular, the usual adjacency matrix was addressed. In the last decade, some types of graphs with very special structures have been proved to be DAS, such as the θ-graphs [25], dumbbell graphs without 4-cycle [32], lollipop graph [3,15], path and its complement [11], etc. See further Refs. [12,19].

Borowiecki [2] pointed out an interesting result: if G_1 and G_2 are bipartite graphs without cycles of length k, $k \equiv 0 \pmod 4$, G_1 and G_2 are per-cospectral if and only if G_1 and G_2 are cospectral. Liu and Zhang in [20,21] investigated systemically which graphs are DPS. They found that graphs determined by the characteristic polynomial are not necessarily determined by the permanental polynomial, and showed that the complete graphs, stars, regular complete bipartite graphs, odd cycles and odd lollipop graphs are DPS. When restricting on connected graphs, the paths, even cycles $C_{4l+2} (l \geq 1)$, lollipop graphs $L_{n,2k+1} (k \geq 1)$ and $L_{n,4}$ are DPS. In addition, Zhang et al. [36] showed that a graph obtained by removing five or fewer edges from a complete graph K_n is DPS, and proved that if $X \subseteq E(K_n)$ induces a star, a matching, or a disjoint union of a matching and a path P_3, then $K_n - X$ is DPS.

In [4], Cámara and Haemers showed that when at most five edges are deleted from K_n, there is just one pair of nonisomorphic cospectral graphs, and constructed nonisomorphic cospectral graphs for all n if six or more edges are deleted from K_n, provided that n is big enough. Motivated by these results, in this paper we intend to investigate when a complete graph K_n with six edges deleted is DPS and DAS respectively.

An outline of this paper is as follows. In Section 2, we shall present some definitions and lemmas, and give a relation between the 4th coefficient of $\pi(G, x)$ and the number of closed walks of length 4 of a graph G. In Section 3, we show that a complete graph with six edges deleted must be DPS. In Section 4, we prove that when six edges are deleted from K_n for $n \geq 7$ and $n = 5$, there is just one pair of non-isomorphic cospectral graphs. We conclude with some discussions about potential applications and future research problems of permanental polynomials of chemical molecular graphs.

2. Preliminaries

For graphs with six edges and isolated vertices, they have at most 12 vertices. We can see that there are exactly five non-isomorphic such graphs with at least 10 vertices. From Appendix I in [16] it can be seen that there exist exactly 63 non-isomorphic such graphs with at most nine vertices. So there exist 68 non-isomorphic graphs with six edges and no isolated vertices. For convenience, let $\mathcal{G}_n$ denote the set of all graphs obtained from K_n by deleting six edges. Thus, up to isomorphism there exist exactly 68 graphs in $\mathcal{G}_n$ for $n \geq 12$, which are labeled by G_i $(i = 1, 2, \ldots, 68)$ and defined as illustrations in Fig. 1.

Let $c_i(G)$ and $p_i(G)$ denote respectively the numbers of i-cycles and i-vertex paths in a graph G. Let $K_3(G)$ be the set of all 3-cycles of G. For each $K \in K_3(G)$, define $d_G(K) = \sum_{v \in V(K)} d_G(v)$ and $D(G) = \sum_{K \in K_3(G)} d_G(K)$. For a subgraph H of G, let $G - E(H)$ be a subgraph obtained from G by deleting the edges of H.

We present the following lemmas which are useful in the proof of the main results.

Lemma 2.1 ([20,36]). *Let G be a graph with n vertices and m edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Suppose that $\pi(G, x) = \sum_{i=0}^{n} b_i(G) x^{n-i}$. Then* (i) $b_0(G) = 1$; (ii) $b_1(G) = 0$; (iii) $b_2(G) = m$; (iv) $b_3(G) = -2c_3(G)$; (v) $b_4(G) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2} + 2c_4(G)$; (vi) $b_5(G) = -2[c_3(G)(m + 3) - D(G) + c_5(G)]$.

Lemma 2.2 ([30,20]). *The following parameters and properties of a graph G can be deduced from the per-spectrum and adjacency spectrum:*

(i) *The number of vertices.*
(ii) *The number of edges.*
(iii) *The number of triangles.*
(iv) *Whether G is bipartite.*
The following can be deduced from the per-spectrum of a graph G:
(v) *The length of a shortest odd cycle.*
(vi) *The number of shortest odd cycles.*
The following can be deduced from the adjacency spectrum of a graph G:
(vii) *Whether G is regular.*
(viii) *Whether G is regular with any fixed girth.*
(ix) *The number of closed walks of any fixed length.*

Lemma 2.3 ([11]). *Let $H \subseteq K_n$ be a graph with l edges and let $G = K_n - E(H)$. Then*

$$c_3(G) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H). \tag{1}$$

Using (1), we can compute the numbers of 3-cycles of all graphs in $\mathcal{G}_n$; see Table 1.