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Some extremal graphs with respect to inverse degree[☆]

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ABSTRACT

The inverse degree of graph G is defined as $ID(G) = \sum_{v \in V(G)} \frac{1}{d_G(v)}$ where $d_G(v)$ is the degree of vertex v in G. In this paper we have determined some upper and lower bounds on the inverse degree $ID(G)$ for a connected graph G in terms of other graph parameters, such as chromatic number, clique number, connectivity, number of cut edges, matching number. Also the corresponding extremal graphs have been completely characterized.

1. Introduction

We only consider finite, undirected and simple graphs throughout this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v (i.e., vertices adjacent to v) in G and $N_G[v] = N_G(v) \bigcup \{v\}$, the *degree* of $v \in V(G)$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. In particular, the maximum and minimum degree of a graph G will be denoted by $\Delta(G)$ and $\delta(G)$, respectively. And a vertex v of degree 1 is called a *pendant vertex* (also known as *leaf*), the edge incident with a pendant vertex is called a *pendant edge*. For a subset W of $V(G)$, let $G-W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, for a subset E' of $E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, the subgraphs $G - W$ and $G - E'$ will be written as $G - v$ and $G - xy$ for short, respectively. For any two nonadjacent vertices x and y of a graph G, we let $G + xy$ be the graph obtained from G by adding an edge xy. Other undefined notations and terminology on the graph theory can be found in [2].

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors such that G can be colored with these colors in order that no two adjacent vertices have the same color. A *clique* of graph G is a subset V_0 of $V(G)$ such that in $G[V_0]$, the subgraph of G induced by V_0, any two vertices are adjacent. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. For two vertex-disjoint graphs G_1 and G_2, we denote by $G_1 \bigcup G_2$ the graph which consists of two components G_1 and G_2. The *join* of G_1 and G_2, denoted by $G_1 \bigvee G_2$, is the graph with vertex set $V(G_1) \bigcup V(G_2)$ and edge set $E(G_1) \bigcup E(G_2) \bigcup \{uv : u \in V(G_1), v \in V(G_2)\}$. Given a graph G, a subset of $V(G)$ is called an *independent set* of G if the subgraph it induces has no edges. The *independence number* of G, denoted by $\alpha(G)$, is defined to be the number of vertices in a largest independent set of G. Two edges e and f in a graph G are called *independent* if they are not incident

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with one common vertex in it. A *matching* of G is a subset of mutually independent edges of G. For a graph G, the *matching number* $\beta(G)$ is the maximum cardinality among the independent sets of edges in G.

For $k \geq 1$, a connected graph G is called *k-connected* if either G is a complete graph K_{k+1} or else it has at least $k+2$ vertices and has no $(k-1)$-vertex cut. Similarly, we call a graph *k-edge-connected* if it has at least two vertices and does not contain $(k-1)$-edge cut. The maximum value of k such that a connected graph G is k-connected is the *connectivity* of G and denoted by $\mathcal{K}(G)$. For a disconnected graph G we define that $\mathcal{K}(G) = 0$. The *edge-connectivity* $\mathcal{K}'(G)$ of a graph G is defined analogously. For a graph G with n vertices, we have the following remarks.

(1) $\mathcal{K}(G) \leq \mathcal{K}'(G) \leq n-1$; (2) $\mathcal{K}(G) = n-1$, $\mathcal{K}'(G) = n-1$ and $G \cong K_n$ are equivalent.

Throughout this paper we use P_n, S_n, C_n and K_n to denote the path graph, star graph, cycle graph and complete graph on n vertices, respectively.

For a graph G, the inverse degree of G is defined as follows:

$$ID(G) = \sum_{v \in V(G)} \frac{1}{d_G(v)}.$$

Maybe the definition of inverse degree of a graph first appeared in the conjectures of computer program [10]. In 2005, Zhang et al. [18] obtained the upper and lower bounds on $ID(T) + \beta(T)$ for any tree T. In 2007, Hu et al. [11] determined the extremal graphs with respect to inverse degree among all connected graphs of order n and with m edges. Dankelmann et al. [4] determined a relation between $ID(G)$ with edge-connectivity of graph. Also Dankelmann et al. [5] provided a bound on diameter in terms of $ID(G)$, which improves a corresponding bound by Erdös et al. [9], Mukwembi [15] presented a better bound on diameter by $ID(G)$ than the above two ones. Especially Li and Shi [13] improved the bound on diameter in terms of $ID(G)$ by Dankelmann et al. [5] for trees and unicyclic graphs. More recently Chen and Fujita [3] obtained a nice relation between diameter and inverse degree of a graph, which settled a conjecture in [15]. Other nice related results can be seen in [7,12,14,17] and the references therein.

In this paper we will present some extremal properties of inverse degree of a graph G. In particular, we determined some upper and lower bounds on inverse degree in terms of chromatic number, clique number, independence number, matching number, (edge-) connectivity, number of cut edges, and characterized the extremal graphs at which the upper or lower bounds are attained.

2. Some lemmas

Before stating our main results, we will list or prove some lemmas as preliminaries, which will play an important role in the next proofs.

For convenience, we define as 0 as the inverse degree of empty graph (i.e., $\overline{K_n}$, the complement of K_n). Correspondingly, when an isolated vertex w is counted in $ID(G)$, the term $\frac{1}{d_G(w)}$ will be viewed as 0.

From the definition of inverse degree, the following lemma can be easily obtained.

Lemma 2.1. *Let G be a graph with $e = xy \in E(G)$ and two nonadjacent vertices $u, v \in V(G)$. Then we have*

(i) *$ID(G - e) > ID(G)$ if $d_G(x) \geq 2$ and $d_G(y) \geq 2$;*
(ii) *$ID(G + uv) < ID(G)$ where neither of vertices u and v is isolated in G.*

Lemma 2.2. *Let G be a connected graph of order $n > 2$ with two vertices $u, v \in V(G)$ such that $d_G(u) \geq d_G(v)$ and $N_G(v) \backslash N_G[u] = \{v_1, v_2, \ldots, v_s\}$ where $s > 0$. A new graph G' is obtained as follows: $G' = (G - \{vv_1, vv_2, \ldots, vv_s\}) + \{uv_1, uv_2, \ldots, uv_s\}$. If $d_G(v) > s$, then we have $ID(G') > ID(G)$.*

Proof. From the structure of the graph G', we find that $d_{G'}(u) = d_G(u) + s$, $d_{G'}(v) = d_G(v) - s$ and all other vertices in G' have the same degrees as those in G. Since $d_G(v) > s$, the vertex v in G' is not isolated. From the definition of inverse degree, we get

$$ID(G') - ID(G) = \frac{1}{d_G(u) + s} + \frac{1}{d_G(v) - s} - \frac{1}{d_G(u)} - \frac{1}{d_G(v)} > 0.$$

Note that the last inequality holds because of the fact that

$$(d_G(u) + s)(d_G(v) - s) = d_G(u)d_G(v) - s(d_G(u) - d_G(v)) - s^2 < d_G(u)d_G(v)$$

when $d_G(u) \geq d_G(v)$. Thus we complete the proof of this lemma. $\square$

The following lemma presents a classic property of concave function. Here, for the completeness, we give a strict proof for it.

Lemma 2.3. *For $x > 0$ let $f(x)$ be a function such that $f''(x) < 0$. Then, for $1 < x_1 \leq x_2$, we have*

$$f(1) + f(x_1 + x_2 - 1) < f(x_1) + f(x_2).$$