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Note

On degree resistance distance of cacti

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ABSTRACT

A graph G is called a cactus if each block of G is either an edge or a cycle. Denote by $Cact(n; t)$ the set of connected cacti possessing n vertices and t cycles. In a recent paper (Du et al., 2015), the $Cact(n; t)$ with minimum degree resistance distance was characterized. We now determine the elements of $Cact(n; t)$ with second-minimum and third-minimum degree resistance distances. In addition, some mistakes in Du et al. (2015) are pointed out.

1. Introduction

Distance is an important concept in graph theory. The ordinary distance $d(u, v) = d_G(u, v)$ between the vertices u and v of the graph G is the length of the shortest path between u and v [5]. For other undefined notations and terminology from graph theory, the readers are referred to [5].

The famous Wiener index W(G) is the sum of distances between all pairs of vertices, i.e., $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v)$. A modified version of the Wiener index is the degree distance defined as $D(G) = \sum_{\{u,v\}\subseteq V(G)}[d(u) + d(v)]d(u, v)$, where $d(u) = d_G(u)$ is the degree of the vertex u of the graph G.

In 1993, Klein and Randić [16] introduced a new distance function named resistance distance, based on the theory of electrical networks. Replacing each edge of a simple connected graph G by a unit resistor. The resistance distance between the vertices u and v of the graph G, denoted by $R(u, v)$, is then defined to be the effective resistance between the nodes u and v in N. If the ordinary distance is replaced by resistance distance in the expression for the Wiener index, one arrives at the Kirchhoff index [7,16]

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R(u, v)$$

which has been widely studied [6,8–12,15]. In 1996, Gutman and Mohar [13] obtained the famous result by which a relationship is established between the Kirchhoff index and the Laplacian spectrum: $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix of a connected graph G with n vertices. For more details on the Laplacian matrix, the readers are referred to [17,18,20,21,24]. Bapat et al. has provided a simple method for computing the resistance distance in [1]. Palacios [28–30,32–34] studied the resistance distance and the Kirchhoff indices of connected

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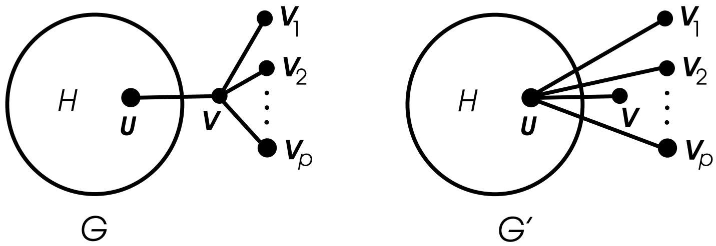

Fig. 1. The σ-transform at v.

undirected graphs with probability methods. E. Bendito et al. [2] formulated the Kirchhoff index based on discrete potential theory. M. Bianchi et al. obtained the upper and lower bounds for the Kirchhoff index $Kf(G)$ of an arbitrary simple connected graph G by using a majorization technique [3,4]. Besides, the Kirchhoff indices of some graphs are investigated in [22,23,25–27,46,47]. The ordinary distance is replaced by resistance distance in the expression for the degree distance, then one arrives at the degree resistance distance [7,12]

$$D_R(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]R(u, v).$$

Palacios [31] named the same graph invariant additive degree-Kirchhoff index.

Tomescu [35] determined the unicyclic and bicyclic graphs with minimum degree distance. In [36], the author investigated the properties of connected graphs having minimum degree distance. Bianchi et al. [3] gave some upper and lower bounds for D_R whose expressions do not depend on the resistance distances. Yang and Klein gave formulae for the degree resistance distances of the subdivisions and triangulations of graphs [44]. For more work on the topological indices, the readers are referred to [14,37,40–43,45].

A graph G is called a cactus if each block of G is either an edge or a cycle. Denote by $Cact(n; t)$ the set of cacti possessing n vertices and t cycles [19,38,39]. In this paper, we determine the second-minimum and third-minimum degree resistance distances among graphs in $Cact(n; t)$ and characterize the corresponding extremal graphs. Also, some mistakes in [3] are pointed out.

2. Preliminaries

Let $R_G(u, v)$ denote the resistance distance between u and v in the graph G. Recall that $R_G(u, v) = R_G(v, u)$ and $R_G(u, v) \geq 0$ with equality if and only if $u = v$.

For a vertex u in G, we define

$$Kf_v(G) = \sum_{u\in G} R_G(u, v) \quad \text{and} \quad D_v(G) = \sum_{u\in G} d_G(u)R_G(u, v).$$

In what follows, for the sake of conciseness, instead of $u \in V(G)$ we write $u \in G$. By the definition of $D_v(G)$, we also have

$$D_R(G) = \sum_{v\in G} d_G(v) \sum_{u\in G} R_G(u, v).$$

Lemma 2.1 ([12]). *Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V(G_1) \cup V(G_2) = V(G)$.*

Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$, $m_2 = |E(G_2)|$. Then

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2Kf_v(G_1) + 2m_1Kf_v(G_2) + (n_2 - 1)D_v(G_1) + (n_1 - 1)D_v(G_2).$$

Definition 2.1 ([7]). Let v be a vertex of degree $p + 1$ in a graph G, such that $vv_1, vv_2, \ldots, vv_p$ are pendent edges incident with v, and u is the neighbor of v distinct from $v_1, v_2, \ldots, v_p$. We form a graph $G' = \sigma(G, v)$ by deleting the edges $vv_1, vv_2, \ldots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$. We say that G' is a σ-transform of the graph G (see Fig. 1).

Lemma 2.2 ([12]). *Let $G' = \sigma(G, v)$ be a σ-transform of the graph G, $d_G(u) \geq 1$. Then $D_R(G) \geq D_R(G')$. Equality holds if and only if G is a star with v as its center.*

Lemma 2.3 ([12]). *Let C_k be the cycle of size k, and $v \in C_k$. Then, $Kf(C_k) = \frac{k^3-k}{12}$, $D_R(C_k) = \frac{k^3-k}{3}$, $Kf_v(C_k) = \frac{k^2-1}{6}$ and $D_v(C_k) = \frac{k^2-1}{3}$.*