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# On the structure and deficiency of $k$-trees with bounded degree

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ABSTRACT

A proper edge colouring of a graph with natural numbers is *consecutive* if colours of edges incident with each vertex form a consecutive interval of integers. The deficiency $def(G)$ of a graph $G$ is the minimum number of pendant edges whose attachment to $G$ makes it consecutively colourable. Since all 1-trees are consecutively colourable, in this paper we study the deficiency of $k$-trees for $k \geq 2$. Our investigation establishes the values of the deficiency of all $k$-trees that have maximum degree bounded from above by $2k$, with $k \in \{2, 3, 4\}$. To obtain these results we consider the structure of $k$-trees with bounded degree and the deficiency of general graphs of odd order. In the first case we show that for $n \geq 2k + 3$ the structure of an $n$-vertex $k$-tree with maximum degree not greater than $2k$ is unique. In the second one we prove that for each $n$-vertex graph $G$ of odd order the inequality $def(G) \geq \frac{1}{2}(|E(G)| - (n-1)\Delta(G))$ holds. Both last mentioned results seem to be interesting in their own right.



## 1. Introduction

The simplest way to define a $k$-tree is by recursion. A $k$-tree of order $k$ is a complete graph and a $k$-tree of order $n, n \geq k+1$, can be obtained by joining a new vertex to any $k$ pairwise adjacent vertices of a $k$-tree of order $n-1$ (the notion introduced by Rose in [20]). Properties of $k$-trees were studied by several authors such as Beineke and Pippert [6], Moon [14], Rose [20] and others.

In this paper we are interested in a class of all $k$-trees with maximum degree bounded from above by $2k$. In Section 3 we characterize this subclass of $k$-trees showing that for $n \geq 2k+3$ each graph of order $n$ from this class has a unique structure (Theorem 2). Moreover, some properties of graphs that have less than $2k+3$ vertices and belong to this class are presented.

In 1987 Asratian and Kamalian [4] began the study on an *interval colouring* of a graph, i.e. a proper edge colouring of a graph with natural numbers such that the colours of edges incident with each vertex form a consecutive interval of integers. Such a colouring have also been called *consecutive*. There are several papers dealing with this topic, but most of them have been concerned bipartite graphs in connection with applications in scheduling theory [12,5,9–11,22,18,3,16,15,17].

Many graphs do not have a consecutive colouring. A simple example is $K_3$. However, every graph, which is not consecutively colourable, can be extended to its supergraph, which has such a colouring, by the attachment of some pendant edges. In Section 4 we study the deficiency of a graph, which is an invariant that measures how close a graph comes to having a consecutive colouring. More precisely, the *deficiency* $def(G)$ of the graph $G$ is the minimum number of pendant edges whose attachment to $G$ makes it consecutively colourable.

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Very little is known about the deficiency of an arbitrary graph. The results from that scope can be found e.g. in [2,1,21]. Our contribution to this topic are Theorems 3 and 5. First of them gives a lower bound on the deficiency of a general graph of odd order. It is the result of some generalization of Giaro et al. result [2], which was connected only with a $\Delta$-regular graph with odd number of vertices. Consequently, Theorem 5 shows the lower bound on the deficiency of a $k$-path of an odd order. We hope that mentioned Theorems 2, 3 and 5 are interesting in their own right. In Section 5 we apply them to calculate the exact values of the deficiency of all $k$-trees with maximum degree bounded from above by $2k$, where $k \in \{2, 3, 4\}$ (Theorems 7, 11 and 15).

## 2. Basic definitions and concepts

In general, we follow the notation and terminology of [8]. The graphs, which we consider are finite, undirected, without loops or multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For any set $S \subseteq V(G)$ the symbol $G[S]$ denotes the subgraph of $G$ induced by $S$. By $deg_G(v)$ and $\Delta(G)$ we denote the degree of a vertex $v$ in a graph $G$ and the maximum degree over all vertices of $G$, respectively. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_G(v)$. A $k$-clique in a graph $G$ is a subset of its $k$ pairwise adjacent vertices. For the simplicity of the notation, if $H$ is a subgraph of $G$, then by $G - H$ we mean a graph with the vertex set $V(G)$ and the edge set $E(G)\backslash E(H)$. We will use this operation only if the graph $G - H$ is uniquely determined up to isomorphism. If graphs $G$ and $H$ are isomorphic, then we write $G \simeq H$. A complete graph of order $n$ is denoted by $K_n$. By $\mathbb{N}$ we denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $a, b \in \mathbb{N}_0$, $a < b$, then the symbol $[a, b]$ denotes the set $\{a, a + 1, \ldots, b - 1, b\}$.

The next definitions are connected with the concept of $k$-trees.

**Definition 1.** The complete graph on $k$ vertices is a $k$-tree. A $k$-tree on $n+1$ vertices ($n \geq k$) can be constructed from a $k$-tree on $n$ vertices by adding a vertex adjacent to all vertices of a $k$-clique of the existing $k$-tree, and only to these vertices.

Let $G$ be a graph on $n$ vertices and $n \geq 2$. For any ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of vertices of $G$ and for every $i$, $2 \leq i \leq n$, we define $N_p(v_i) = N_G(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$. Let $k \leq n$. An ordering $\mathbf{v} = (v_1, \ldots, v_n)$ we call the *$k$-tree ordering* of the graph $G$ if $G[\{v_1, \ldots, v_k\}] \simeq K_k$ and $G[N_p(v_i)] \simeq K_k$, for every $i$ such that $k + 1 \leq i \leq n$. Note that $G$ on $n$ vertices is a $k$-tree if and only if it has a $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$. If there exists a $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ such that $N_p(v_i) = \{v_{i-s} : 1 \leq s \leq k\}$, for every $i$ such that $k + 1 \leq i \leq n$, then such a graph $G$ we call the *$k$-path* and denote by $P_n^k$. Note that $P_n^k$ is unique up to isomorphism, for all permissible $k$ and $n$. It should be mentioned here that our definition of a $k$-path is different from the definitions of a *$k$-path of a graph* introduced by Proskurowski in [19] and a *$k$-path* introduced by Markenzon et al. in [13]. For any $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ and any $m$, $1 \leq m \leq n$, we define $G_m^{\mathbf{v}} = G[\{v_1, \ldots, v_m\}]$.

**Definition 2.** For any $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ the linearity $l(\mathbf{v}, G)$ of $\mathbf{v}$ is

$$\max\left\{r \in \mathbb{N} : \forall_{2 \leq i \leq r}\ N_p(v_i) = \{v_{i-s} : 1 \leq s \leq \min\{k, i - 1\}\}\right\}.$$

**Definition 3.** A $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ is the base $k$-tree ordering if

$$l(\mathbf{v}, G) = \max\left\{l(\mathbf{v}', G) : \mathbf{v}' \text{is a } k\text{-tree ordering of the graph } G\right\}.$$

From the last definitions we immediately obtain the following remarks.

**Remark 1.** A base $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ is not uniquely determined.

**Remark 2.** For any base $k$-tree ordering $\mathbf{v} = (v_1, \ldots, v_n)$ of the graph $G$ the linearity $l(\mathbf{v}, G)$ is an order of the longest $k$-path included in $G$.

**Remark 3.** If $\mathbf{v} = (v_1, \ldots, v_n)$ is a $k$-tree ordering of the graph $G$, then every $k$-clique of the graph $G_m^{\mathbf{v}}$, where $k + 1 \leq m \leq l(\mathbf{v}, G)$, consists of any $k$ from among any $k + 1$ consecutive vertices of $(v_1, \ldots, v_m)$.

**Remark 4.** If $\mathbf{v} = (v_1, \ldots, v_n)$ is a base $k$-tree ordering of the graph $G$, then $l(\mathbf{v}, G) \geq \min\{k + 2, n\}$. Moreover, $G$ is isomorphic to $P_n^k$ if and only if $l(\mathbf{v}, G) = n$.

## 3. The structure of $k$-trees with bounded degree

In this section we characterize the structure of $k$-trees with maximum degree bounded from above by $2k$.

**Lemma 1.** *Let $\mathbf{v} = (v_1, \ldots, v_n)$ be a base $k$-tree ordering of the graph $G$ and let $\Delta(G) \leq 2k$. If $l(\mathbf{v}, G) \leq n - 1$, then $l(\mathbf{v}, G) \leq 2k + 1$.*