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A proof of the conjecture regarding the sum of domination number and average eccentricity

Zhibin Du [a,*], Aleksandar Ilić [b]

[a] School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, China

[b] Mathematical Institute, Serbian Academy of Science and Arts, Knez Mihajlova 36, 11000 Belgrade, Serbia

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ABSTRACT

The average eccentricity $ecc(G)$ of a graph G is the mean value of eccentricities of all vertices of G. In this paper, we continue the work of Ilić (2012) and resolve a conjecture, obtained by the system AutoGraphiX, about the upper bound on the sum of the average eccentricity and the domination number among connected graphs on n vertices.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$. The closed neighborhood of a vertex v in a graph G is the vertex subset of G each of which is either v or a neighbor of v in G.

For vertices $u, v \in V(G)$, the distance $d_G(u, v)$ is defined as the length of a shortest path between u and v in G. The eccentricity of a vertex is the maximum distance from it to any other vertex,

$$\varepsilon_G(v) = \max_{u \in V(G)} d_G(u, v).$$

The average eccentricity of a graph G is the mean value of eccentricities of all vertices of G,

$$ecc(G) = \frac{1}{n} \sum_{v \in V(G)} \varepsilon_G(v).$$

In theoretical chemistry molecular structure descriptors (also called topological indices) are used for modeling physico-chemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. There exist several types of such indices, especially those based on the distances, such as the Wiener index $W(G)$ [5] or the eccentric connectivity index [8,10–12,15].

Dankelmann, Goddard and Swart [3] presented some upper bounds and formulas for the average eccentricity regarding the diameter and the minimum vertex degree. Recently, Dankelmann and Mukwembi [4] established several upper bounds on the average eccentricity in terms of the independence number, chromatic number, domination number and connected domination number, respectively.

* Corresponding author.
E-mail addresses: zhibindu@126.com (Z. Du), aleksandari@gmail.com (A. Ilić).

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Trees and unicyclic graphs are the simplest connected graphs, their average eccentricities are of particular interest. Tang and Zhou in [13] presented some lower and upper bounds for the average eccentricity of trees with fixed diameter, fixed number of pendent vertices and fixed matching number, respectively, and determined the n-vertex trees with the first four smallest and the first $n/2$-th largest average eccentricities for $n \geq 6$, and later they [14] determined the n-vertex unicyclic graphs with the first $(\lfloor \frac{n}{2} \rfloor - 1)$-th largest average eccentricities for $n \geq 6$.

AutoGraphiX (AGX) [1,2] is an interactive software designed to help to find conjectures in graph theory developed by GERAD group from Montréal. It uses the variable neighborhood search metaheuristic and data analysis methods to find extremal graphs with respect to one or more invariants. Recently there are vast researches regarding AGX conjectures and a series of papers on various graph invariants. In this paper, we continue this work and resolve conjectures from [6] and [9].

A dominating set of a graph is a vertex subset whose closed neighborhood contains all vertices of the graph. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [7].

Ilić in [9] analyzed the extremal properties of the average eccentricity, by introducing two graph transformations that increase/decrease $ecc(G)$. Furthermore, the authors of this paper [6] resolved five conjectures, obtained by the system AutoGraphiX, about the average eccentricity and other graph parameters (the chromatic number, the Randić index and the independence number), and refuted two conjectures about the average eccentricity and the spectral radius.

It is worth mentioning that Ilić [9] corrected one conjecture about the average eccentricity and the domination number (Conjecture A.464-L from [1]) — which is the main motivation of this paper.

Let P_n be the path on n vertices. Let D_n be the n-vertex tree obtained from a path $P_{n-1} = v_1 v_2 \cdots v_{n-1}$ by attaching a pendent vertex to v_3, where $n \geq 6$.

Conjecture 1.1. *Let G be an n-vertex connected graph, where $n \geq 4$. Then*

$$\gamma(G) + ecc(G) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil + \frac{1}{n} \left\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \right\rfloor & \text{if } n \not\equiv 0 \pmod 3 \\ \frac{n}{3} + 2 - \frac{3}{n} + \frac{1}{n} \left\lfloor \frac{3}{4}(n-1)^2 - \frac{1}{2}(n-1) \right\rfloor & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

with equality if and only if $G \cong P_n$ when $n \not\equiv 0 \pmod 3$ and $G \cong D_n$ when $n \equiv 0 \pmod 3$.

Note that the lower bound of $\gamma(G) + ecc(G)$ can be obtained easily, i.e.,

$$\gamma(G) + ecc(G) \geq 2$$

with equality if and only if $G \cong K_n$, where K_n is the complete graph on n vertices.

The paper is organized as follows. In Section 2 we introduce two graph transformations that increase the sum $\gamma(G) + ecc(G)$. In Section 3, we analyze the sum of domination number and average eccentricity of caterpillars by considering four different cases. In Section 4, we finally solve Conjecture 1.1.

2. Two transformations

For an edge subset M of the graph G, let $G - M$ denote the graph obtained from G by deleting the edges in M, and for an edge subset M^* of the complement of G, let $G + M^*$ denote the graph obtained from G by adding the edges in M^*. In particular, if $M = \{uv\}$, then we write $G - uv$ for $G - \{uv\}$, and if $M^* = \{uv\}$, then we write $G + uv$ for $G + \{uv\}$.

For a vertex subset N of the graph G, let $G - N$ denote the graph obtained from G by deleting the vertices in N and their incident edges. In particular, if $N = \{v\}$, then we write $G - v$ for $G - \{v\}$.

For a set A and a vertex v, if $v \in A$, then we write $A - \{v\}$ to represent $A \setminus \{v\}$, and if $v \notin A$, then we write $A + \{v\}$ to represent $A \cup \{v\}$.

A diametrical path of a graph is a shortest path whose length is equal to the diameter of the graph.

Suppose that v is a pendent vertex in G with unique neighbor u. Let A be a dominating set of G of order $\gamma(G)$, where $v \in A$. By the minimality of A, we know that $u \notin A$. Let $A^* = A - \{v\} + \{u\}$. Clearly, A^* is also a dominating set of G, and $|A| = |A^*| = \gamma(G)$. Thus we may assume that the dominating sets of order $\gamma(G)$ considered in our proofs contain no pendent vertex of G.

First we present two transformations which will be of great importance in our proofs.

Lemma 2.1. *Let H be a nontrivial tree, where $u, v \in V(H)$. Let G_1 be the tree obtained from H by attaching two pendent vertices w_1 and w_2 to u.*

(i) *For $G_2 = G_1 - uw_1 + w_2 w_1$, see Fig. 1, we have*

$$\gamma(G_1) + ecc(G_1) < \gamma(G_2) + ecc(G_2).$$

(ii) *Suppose that v has a pendent neighbor v_1 in H. Let $G_3 = G_2 - uw_2 + vw_2$, see Fig. 2. If there is a diametrical path P of G_2 such that $v, v_1 \in V(P)$, $w_1, w_2 \notin V(P)$, then*

$$\gamma(G_2) + ecc(G_2) < \gamma(G_3) + ecc(G_3).$$