Discrete Applied Mathematics 201 (2016) 130–140

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Almost every graph is divergent under the biclique operator

Marina Groshaus [a], André L.P. Guedes [a,b], Leandro Montero [a,c,*]

[a] Universidad de Buenos Aires, Departamento de Computación, Argentina
[b] Universidade Federal do Paraná, Departamento de Informática, Brazil
[c] Université Paris-Sud, Laboratoire de Recherche en Informatique, France

ARTICLE INFO

Article history:
Received 10 November 2014
Received in revised form 26 June 2015
Accepted 18 July 2015
Available online 24 August 2015

Keywords:
Bicliques
Biclique graphs
False-twin-free graphs
Iterated graph operators
Graph dynamics

ABSTRACT

A biclique of a graph G is a maximal induced complete bipartite subgraph of G. The biclique graph of G denoted by $KB(G)$, is the intersection graph of all the bicliques of G. The biclique graph can be thought as an operator between the class of all graphs. The iterated biclique graph of G denoted by $KB^k(G)$, is the graph obtained by applying the biclique operator k successive times to G. The associated problem is deciding whether an input graph converges, diverges or is periodic under the biclique operator when k grows to infinity. All possible behaviors were characterized recently and an $O(n^4)$ algorithm for deciding the behavior of any graph under the biclique operator was also given. In this work we prove new structural results of biclique graphs. In particular, we prove that every false-twin-free graph with at least 13 vertices is divergent. These results lead to a linear time algorithm to solve the same problem.

1. Introduction

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. For example, line graphs (intersection graphs of the edges of a graph), interval graphs (intersection of intervals of the real line), clique graphs (intersection of cliques of a graph), etc. [4,5,8,12,13,27,29].

The *clique graph* of G denoted by $K(G)$, is the intersection graph of the family of all maximal cliques of G. Clique graphs were introduced by Hamelink in [19] and characterized by Roberts and Spencer in [35]. The computational complexity of the recognition problem of clique graphs had been open for more than 40 years. In [1] they proved that clique graph recognition problem is NP-complete.

The clique graph can be thought as an operator between the class of all graphs. The *iterated clique graph* $K^k(G)$ is the graph obtained by applying the clique operator k successive times ($K^0(G) = G$). Then K is called *clique operator* and it was introduced by Hedetniemi and Slater in [20]. Much work has been done on the scope of the clique operator looking at the different possible behaviors. The associated problem is deciding whether an input graph converges, diverges or is periodic under the clique operator when k grows to infinity. In general it is not clear that the problem is decidable. However, partial characterizations have been given for convergent, divergent and periodic graphs restricted to some classes of graphs. Some of these lead to polynomial time recognition algorithms. For the clique-Helly graph class, graphs which converge to the trivial graph have been characterized in [3]. Cographs, P_4-tidy graphs, and circular-arc graphs are examples of classes where the different behaviors are characterized [7,30]. Divergent graphs were also considered. For example in [22], families of divergent graphs are shown. Periodic graphs were studied in [8,26]. In particular it is proved that for every integer i, there

* Corresponding author at: Université Paris-Sud, Laboratoire de Recherche en Informatique, France.
E-mail addresses: groshaus@dc.uba.ar (M. Groshaus), andre@inf.ufpr.br (A.L.P. Guedes), lmontero@lri.fr, lmontero@dc.uba.ar (L. Montero).

http://dx.doi.org/10.1016/j.dam.2015.07.022
0166-218X/

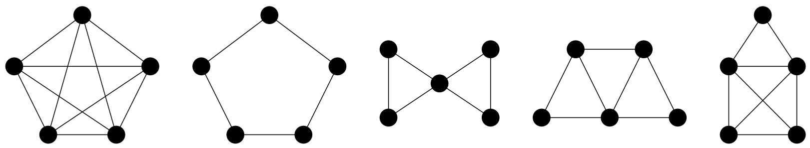

Fig. 1. Graphs K_5, C_5, *butterfly*, *gem* and *rocket*, respectively.

exist periodic graphs with period *i* and also convergent graphs which converge in *i* steps. More results about iterated clique graph can be found in [11,23–25,32,33].

A biclique is a maximal bipartite complete induced subgraph. Bicliques have applications in various fields, for example biology: protein–protein interaction networks [6], social networks: web community discovery [21], genetics [2], medicine [31], information theory [18], etc. More applications (including some of these) can be found in [28].

The *biclique graph* of a graph *G* denoted by $KB(G)$, is the intersection graph of the family of all maximal bicliques of *G*. It was defined and characterized in [16]. However no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construction can be viewed as an operator between the class of graphs.

The *iterated biclique graph* $KB^k(G)$ is the graph obtained by applying to *G* the biclique operator *KBk* times iteratively. It was introduced in [14] and all possible behaviors were characterized. It was proven that a graph *G* is either divergent or convergent but it is never periodic (with period bigger than 1). In addition, general characterizations for convergent and divergent graphs were given. These results were based on the fact that if a graph *G* contains a clique of size at least 5, then $KB(G)$ or $KB^2(G)$ contains a clique of larger size. Therefore, in that case *G* diverges. Similarly if *G* contains the *gem* or the *rocket* graphs as an induced subgraph, then $KB(G)$ contains a clique of size 5, and again *G* diverges. Otherwise it was shown that after removing false-twin vertices of $KB(G)$, the resulting graph is a clique on at most 4 vertices, in which case *G* converges. Moreover, it was proved that if a graph *G* converges, it converges to the graphs K_1 or K_3, and it does so in at most 3 steps. These characterizations leaded to an $O(n^4)$ time algorithm for recognizing convergent or divergent graphs under the biclique operator.

In this work we show new results that lead to a linear time algorithm to solve the same problem. We study conditions for a graph to contain a K_5, a C_5, a *butterfly*, a *gem* or a *rocket* (see Fig. 1) as induced subgraphs so that we can decide divergence (since $K_5 \subseteq KB(C_5)$, $KB(butterfly)$, $KB(gem)$, $KB(rocket)$). First we prove that if *G* has at least 7 bicliques then it diverges. Then, we show that every false-twin-free graph with at least 13 vertices has at least 7 bicliques, and therefore diverges. Since adding false-twins to a graph does not change its *KB* behavior, then the linear algorithm is based on the deletion of false-twin vertices of the graph and looking at the size of the remaining graph.

It is worth to mention that these results are indeed very different from the ones known for the clique operator, for which it is still an open problem to know the computational complexity of deciding the behavior of a graph under the clique operator.

This work is the full version of a previous extended abstract published in [15]. It is organized as follows. In Section 2 the notation is given. Section 3 contains some preliminary results that we will use later. In Section 4 we prove that any graph with at least 7 bicliques diverges, and that every graph with at least 13 vertices with no false-twins vertices contains at least 7 bicliques. This leads to a linear time algorithm to decide convergence or divergence under the biclique operator.

2. Notation and terminology

Along the paper we restrict to undirected simple graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $n = |V(G)|$ and $m = |E(G)|$. A *subgraph* G' of *G* is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. A subgraph $G' = (V', E')$ of *G* is *induced* when for every pair of vertices $v, w \in G'$, $vw \in E'$ if and only if $vw \in E$. A graph *G* is *H-free* if it does not contain *H* as an induced subgraph. A graph $G = (V, E)$ is *bipartite* when $V = U \cup W$, $U \cap W = \emptyset$ and $E \subseteq U \times W$. Say that *G* is a *complete graph* when every possible edge belongs to *E*. A complete graph of *n* vertices is denoted K_n. A *clique* of *G* is a maximal complete induced subgraph while a *biclique* is a maximal bipartite complete induced subgraph of *G*. The *open neighborhood* of a vertex $v \in V(G)$ denoted $N(v)$, is the set of vertices adjacent to v while the *closed neighborhood* of v denoted by $N[v]$, is $N(v) \cup \{v\}$. Two vertices u, v are *false-twins* if $N(u) = N(v)$. A vertex $v \in V(G)$ is *universal* if it is adjacent to all of the other vertices in $V(G)$. A *path* (*cycle*) of *k* vertices, denoted by P_k (C_k), is a set of vertices $v_1 v_2 \dots v_k \in V(G)$ such that $v_i \neq v_j$ for all $1 \leq i \neq j \leq k$ and v_i is adjacent to v_{i+1} for all $1 \leq i \leq k-1$ (and v_1 is adjacent to v_k). A graph is *connected* if there exists a path between each pair of vertices. We assume that all the graphs of this paper are connected.

A *rocket* is a complete graph with 4 vertices and a vertex adjacent to two of them. A *butterfly* is the graph obtained by joining two copies of the K_3 with a common vertex.

Given a family of sets $\mathcal{H}$, the *intersection graph* of $\mathcal{H}$ is a graph that has the members of $\mathcal{H}$ as vertices and there is an edge between two sets $E, F \in \mathcal{H}$ when *E* and *F* have non-empty intersection.

A graph *G* is an *intersection graph* if there exists a family of sets $\mathcal{H}$ such that *G* is the intersection graph of $\mathcal{H}$. We remark that any graph is an intersection graph [37].

A family of sets $\mathcal{H}$ is *mutually intersecting* if every pair of sets $E, F \in \mathcal{H}$ have non-empty intersection.

Let *F* be any graph operator. Given a graph *G*, the *iterated graph* under the operator *F* is defined iteratively as follows: $F^0(G) = G$ and for $k \geq 1$, $F^k(G) = F^{k-1}(F(G))$. We say that a graph *G* *diverges* under the operator *F* whenever $\lim_{k\to\infty}$