Discrete Applied Mathematics 201 (2016) 150–163

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Convex hulls of superincreasing knapsacks and lexicographic orderings

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ARTICLE INFO

Article history:
Received 26 August 2013
Received in revised form 6 March 2015
Accepted 8 August 2015
Available online 29 August 2015

Keywords:
Superincreasing sequence
Lexicographic ordering
Greedy solution knapsack
Convex hull
Linear time complexity

ABSTRACT

We consider bounded integer knapsacks where the weights and variable upper bounds together form a superincreasing sequence. The elements of this superincreasing knapsack are exactly those vectors that are lexicographically smaller than the greedy solution to optimizing over this knapsack. We describe the convex hull of this n-dimensional set with $\mathcal{O}(n)$ facets. We also establish a distributive property by proving that the convex hull of $\leq$- and $\geq$-type superincreasing knapsacks can be obtained by intersecting the convex hulls of $\leq$- and $\geq$-sets taken individually. Our proofs generalize existing results for the $0\backslash 1$ case.
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1. Introduction

Given positive integers n, b, and (a_i, u_i) for all $i \in N := \{1, \dots, n\}$, we consider a bounded integer knapsack defined as $K := \{x \in \mathbb{Z}^n_+ : a^\top x \leq b,\ 0 \leq x_i \leq u_i\ i = 1, \dots, n\}$. Without loss of generality (w.o.l.o.g.) we assume that $a_i u_i \leq b\ \forall i \in N$ and that $a^\top u > b$ to ensure a nontrivial set. When all the upper bounds are equal to one, we have the $0\backslash 1$ knapsack K^1. The convex hull of K, denoted by $\operatorname{conv} K$, is referred to as the knapsack polytope.

The study of the knapsack polytope has received considerable attention in literature and in general, there may exist exponentially many facet-defining inequalities. There exist special classes of K for which a complete description of $\operatorname{conv} K$ is known. For the $0\backslash 1$ knapsack, these results include the minimal covers of Wolsey [25] assuming certain matroidal properties for $a^\top x \leq b$, $(1, k)$-configurations of Padberg [18], weight-reduction principle of Weismantel [24] when $a_i \in \{1, \lfloor b/3 \rfloor + 1, \lfloor b/3 \rfloor + 2, \dots, \lfloor b/2 \rfloor\}\ \forall i$ or $a_i \in \{1, \lfloor b/2 \rfloor + 1, \lfloor b/2 \rfloor + 2, \dots, b\}\ \forall i$, and Weismantel [23] when $a_i \in \{\acute{a}, \breve{a}\}\ \forall i$ and for two distinct positive integers $\acute{a}$ and $\breve{a}$. There also exist complete descriptions of $\operatorname{conv} K$ for upper bounds not equal to 1. For a *divisible* knapsack, i.e. when $a_{i-1} \mid a_i$ for all $i \geq 2$, three results are known: (i) Marcotte [13] when $K = \{x \in \mathbb{Z}^n_+ : a^\top x \leq b\}$, (ii) Pochet and Wolsey [21] when $K = \{x \in \mathbb{Z}^n_+ : a^\top x \geq b\}$, and (iii) Pochet and Weismantel [20] when $K = \{x \in \mathbb{Z}^n_+ : a^\top x \leq b, \mathbf{0} \leq x \leq u\}$. Recently, Cacchiani et al. [5] described the convex hulls of $\leq$- and $\geq$-type knapsacks with a generalized upper bound constraint $\sum_{i \in N} x_i \leq 2$. The polytopes in [20,21] involve an exponential number of valid inequalities, whereas the polytopes in [5,13] have $\mathcal{O}(n)$ facets.

In this paper, we are interested in the convex hull of a special class of K characterized as follows.

Definition 1 (*Superincreasing Knapsack*)**.** The set K is said to be a *superincreasing knapsack* if $\{(a_i, u_i)\}_{i \in N}$ forms a weakly superincreasing sequence of tuples, i.e. $\sum_{k=1}^{i} a_k u_k \leq a_{i+1}\ \forall i \geq 1$.

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http://dx.doi.org/10.1016/j.dam.2015.08.010
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$0\backslash 1$ superincreasing knapsacks have been historically used in cryptographic systems [15,17]; their structure and linear time complexity were investigated in [22]. Atkinson et al. [3] obtained a wider class of $0\backslash 1$ knapsacks with linear time complexity. From the viewpoint of a polyhedral study, the knapsack polytope for superincreasing K^1 was first studied by Laurent and Sassano [10], whose result is paraphrased below.

Theorem 1 *([10]). For any positive integer b, the* $0\backslash 1$ *knapsack polytope* $\operatorname{conv} K^1[n][b]$ *is completely described by its minimal cover inequalities if and only if* $\{(a_i, 1)\}_{i\in N}$ *forms a weakly superincreasing sequence. Furthermore, all the* $\mathcal{O}(n)$ *minimal covers can be explicitly enumerated.*

We extend the sufficiency condition of Theorem 1 to the case when the variable upper bounds in K are not necessarily equal to one. We explicitly describe the convex hull with $\mathcal{O}(n)$ nontrivial facets. As is to be expected, the proposed inequalities reduce to minimal covers of K^1 when u is a vector of ones. The convex hull proof for K^1 is simpler since the coefficient matrix for system of minimal covers is an interval matrix and hence totally unimodular. Since Marcotte [13] describes the convex hull of divisible K using $\mathcal{O}(n)$ facets, it is obvious that the superincreasing property is not a necessary condition for $\operatorname{conv} K$ to have $\mathcal{O}(n)$ facets. Besides generalizing the result of Laurent and Sassano, another motivation for studying superincreasing knapsacks is that such sets appear after reformulating the integer variables in a mixed integer program; see Gupte et al. [9]. The most common example of such reformulations is the set

$$K_\alpha(b) := \left\{\zeta :\ \sum_t \alpha^{t-1}\zeta_t \le u,\ \zeta_t \in \{0,1,\ldots,\alpha-1\}\ \forall t = 1,\ldots,\lfloor \log_\alpha u_+ \rfloor 1\right\} \tag{1}$$

obtained after α-nary expansion of an integer variable: $x = \sum_{t=1}^{\lfloor \log_\alpha u_i \rfloor + 1} \alpha^{t-1}\zeta_t$. Convex hull of $K_2(b)$ was independently studied by [8,9]. A complete knowledge of the superincreasing knapsack polytope will provide a family of valid inequalities to the mixed integer program. Gupte et al. demonstrated the practical usefulness of facets to binary expansion knapsacks as cutting planes in a branch-and-cut algorithm for solving mixed integer bilinear programs.

Remark 1. The extended formulation of K, obtained after adding new variables $\zeta_{it} \in \{0, 1\}\ \forall i, t$ and basis expansion of each x_i as $x_i = \sum_{t=1}^{\lfloor \log_\alpha u_i \rfloor + 1} \alpha^{t-1}\zeta_{it}$ for some $\alpha \in \mathbb{Z}_{++}$, does not obey the superincreasing property. Hence we cannot obtain $\operatorname{conv} K$ simply as a projection of the extended formulation.

Note that there is no inclusive relationship between superincreasing and divisible knapsacks. However, certain types of knapsacks, such as $K_\alpha(b)$, may be both divisible and superincreasing. For divisible superincreasing knapsacks, our result provides an explicit linear size minimal description as compared to the implicit exponential size description in [20].

Throughout this paper, we assume that K is superincreasing. We begin by analyzing the greedy solution of K in Section 2 and use it to provide a useful geometric interpretation to our assumption of superincreasing tuples $\{(a_i, u_i)\}_{i\in N}$. Section 3 derives a set of facet-defining inequalities, referred to as *packing inequalities*, to $\operatorname{conv} K$ and our first main result in Theorem 2 proves that these inequalities describe $\operatorname{conv} K$. In Section 4, we prove that the convex hull of intersection of two superincreasing knapsacks is given by the facets of the individual knapsack polytopes. This second main result in Theorem 3 is indeed interesting since the convex hull operator does not distribute in general and further implies that the convex hull of a family of m intersecting superincreasing knapsacks is described by $\mathcal{O}(n)$ linear inequalities. For general $0\backslash 1$ knapsacks, new valid inequalities were derived in [11,14] for intersection of two $\le$-type knapsacks and in [7,11] for one $\le$- and one $\ge$-type knapsack.

We adopt the following notation. $\operatorname{conv} \mathcal{X}$ is the convex hull of a set $\mathcal{X}$. $\mathbb{Z}_+(\mathbb{Z}_{++})$ is the set of nonnegative (positive) integers. $\mathbf{e}$ is a vector of ones, $\mathbf{e}_i$ is the ith unit vector and $\mathbf{0}$ is a vector of zeros. $\mathcal{H}_n(u) := \{x \in \mathbb{Z}^n_+ : x_i \le u_i\ \forall i\}$ is a discrete hyper-rectangle. For $\ell > s$, we denote $\sum_\ell^s(\cdot) = 0$ and $\prod_\ell^s(\cdot) = 1$. The positive part of $\xi \in \Re$ is denoted by $[\xi]^+ := \max\{0, \xi\}$.

2. Structure of *K*

This section discusses structural properties of a superincreasing knapsack—first we present a geometric interpretation to the algebraic requirements of Definition 1, then we characterize maximal packings of K and finally we state a dynamic program to optimize over K. The proposed results, especially the maximal packing and dynamic program, are known in literature for the $0\backslash 1$ case, see for example Shamir [22], which has important applications in cryptographic systems. Our contribution is to extend these results to the general integer case and establish a foundation for our main theorems in Sections 3 and 4.

The notion of *lexicographic ordering* will be useful for the rest of the paper. For any two vectors v^1 and v^2, the vector v^1 is lexicographically smaller than v^2, denoted as $v^1 \preccurlyeq v^2$, if either $v^1 = v^2$ or the first (in reverse order) nonzero element i of $v^1 - v^2$ is such that $v^1_i < v^2_i$. In the latter case, we denote $v^1 \prec v^2$. Since $\preccurlyeq$ is a total order, for any distinct v^1 and v^2, either $v^1 \prec v^2$ or $v^2 \prec v^1$ (equivalently $v^1 \succ v^2$).

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