Discrete Applied Mathematics 201 (2016) 164–171

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Semi-transitive orientations and word-representable graphs[☆]

Magnús M. Halldórsson [a], Sergey Kitaev [b,*], Artem Pyatkin [c]

[a] School of Computer Science, Reykjavik University, 101 Reykjavik, Iceland
[b] Department of Computer and Information Sciences, University of Strathclyde, Glasgow G1 1XH, UK
[c] Sobolev Institute of Mathematics, Novosibirsk State University, Novosibirsk, Russia

ARTICLE INFO

Article history:
Received 28 January 2015
Received in revised form 27 July 2015
Accepted 31 July 2015
Available online 24 August 2015

Keywords:
Graphs
Words
Orientations
Word-representability
Complexity
Circle graphs
Comparability graphs

ABSTRACT

A graph $G = (V, E)$ is a *word-representable graph* if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$.

In this paper we give an effective characterization of word-representable graphs in terms of orientations. Namely, we show that a graph is word-representable if and only if it admits a *semi-transitive orientation* defined in the paper. This allows us to prove a number of results about word-representable graphs, in particular showing that the recognition problem is in NP, and that word-representable graphs include all 3-colorable graphs.

We also explore bounds on the size of the word representing the graph. The representation number of G is the minimum k such that G is a representable by a word, where each letter occurs k times; such a k exists for any word-representable graph. We show that the representation number of a word-representable graph on n vertices is at most $2n$, while there exist graphs for which it is $n/2$.

1. Introduction

A graph $G = (V, E)$ is *word-representable* if there exists a word W over the alphabet V such that for each pair of distinct letters x and y, $(x, y) \in E$ if and only if the occurrences of the letters alternate in W. As an example, the words *abcdabcd*, *abcddcba*, and *abdacdbc* represent the 4-clique, K_4; 4-independent set, $\overline{K_4}$; and the 4-cycle, C_4, labeled by a, b, c, d in clockwise direction, respectively.

If each letter appears exactly k times in the word, the word is said to be *k-uniform* and the graph is said to be *k-word-representable*. It is known that any word-representable graph is k-word-representable, for some k [12].

The class of word-representable graphs is rich, and properly contains several important graph classes to be discussed next.

Circle graphs. Circle graphs are those whose vertices can be represented as chords on a circle in such a way that two nodes in the graph are adjacent if and only if the corresponding chords overlap. Assigning a letter to each chord and listing the letters in the order they appear along the circle, one obtains a word where each letter appears twice and two nodes are adjacent if and only if the letter occurrences alternate [4]. Therefore, circle graphs are the same as 2-word-representable graphs.

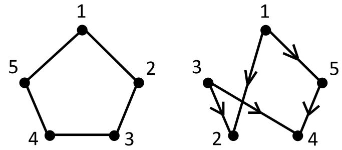
☆ A preliminary version of this work appeared in the International Workshop on Graph-Theoretic Concepts in Computer Science (WG), 2008.

* Corresponding author.
E-mail addresses: mmh@ru.is (M.M. Halldórsson), sergey.kitaev@cis.strath.ac.uk (S. Kitaev), artem@math.nsc.ru (A. Pyatkin).

http://dx.doi.org/10.1016/j.dam.2015.07.033

3125145342

(a) Task execution sequence.



(b) Word-representable graph.

(c) Semi-transitive orientation.

Fig. 1. The word in (a) corresponds to the word-representable graph in (b). A semi-transitive orientation of the graph is given in (c).

Comparability graphs. A comparability graph is one that admits a transitive orientation of the edges, i.e. an assignment of directions to the edges such that the adjacency relation of the resulting digraph is transitive: the existence of arcs xy and yz yields that xz is an arc. Such a digraph induces a poset on the set of vertices V. Note that each poset is an intersection of several linear orders and each linear order corresponds to some permutation P_i of V. These permutations can be concatenated to a word of the form $P_1P_2 \cdots P_k$. Then two letters alternate in this word if and only if they are in the same order in each permutation (linear order), and this means that they are comparable in the poset and, thus, the corresponding letters are adjacent in the graph. So, comparability graphs are a subclass of word-representable graphs that is known as the class of *permutationally representable graphs* in the literature.

Cover graphs. The (*Hasse*) *diagram* of a partial order $P = (V, <)$ is the directed graph on V with an arc from x to y if $x < y$ and there is no z with $x < z < y$ (in which case x "covers" y). A graph is a *cover graph* if it can be oriented as a diagram of a partial order. Limouzy [14] observed that cover graphs are exactly the triangle-free word-representable graphs.

3-colorable graphs. A corollary of our main structural result in this paper is that the class of word-representable graphs contains all 3-colorable graphs.

Various computational hardness results follow from these inclusions. Most importantly, since it is an NP-hard problem to recognize cover graphs [1,16], the same holds for word-representable graphs. Also, the NP-hardness of various optimization problems, such as Independent Set, Dominating Set, Graph Coloring, and Clique Partition, follows from the case of 3-colorable graphs.

Our results. The main result of the paper is an alternative characterization of word-representable graphs in terms of orientations.

A directed graph (digraph) $G = (V, E)$ is *semi-transitive* if it is acyclic and for any directed path $v_1v_2 \cdots v_k$, either $v_1v_k \notin E$ or $v_iv_j \in E$ for all $1 \le i < j \le k$. Clearly, comparability graphs (i.e. those admitting transitive orientations) are semi-transitive. The main result of this paper is Theorem 3 saying that a graph is word-representable if and only if it admits a semi-transitive orientation.

The proof of the main result shows that any word-representable graph on $n \ge 3$ vertices is $(2n-4)$-word-representable. This bound implies that the problem of recognizing word-representable graphs is contained in NP. Previously, no polynomial upper bound was known on the *representation number*, the smallest value k such that the given graph is k-word-representable. Our bound on the representation number is tight up to a constant factor, as we construct graphs with representation number $n/2$. We also show that deciding if a word-representable graph is k-word-representable is NP-complete for $3 \le k \le n/2$.

One corollary of the structural result is that all 3-colorable graphs are word-representable. This provides a generic reason for word-representability for some of the classes of graphs previously known to be word-representable, e.g. for outerplanar graphs and prisms. On the other hand, there are non-3-colorable graphs that are word-representable (for example, any complete graph on at least four vertices).

A motivating application. Consider a scenario with n recurring tasks with requirements on the alternation of certain pairs of tasks. This captures typical situations in periodic scheduling, where there are recurring *precedence* requirements.

When tasks occur only once, the pairwise requirements form precedence constraints, which are modeled by partial orders. When the orientation of the constraints is omitted, the resulting pairwise constraints form comparability graphs. The focus of this paper is to study the class of undirected graphs induced by the alternation relationship of recurring tasks.

Consider, e.g. the following five tasks that may be involved in operation of a given machine: (1) Initialize controller, (2) Drain excess fluid, (3) Obtain permission from supervisor, (4) Ignite motor, (5) Check oil level. Tasks 1 & 2, 2 & 3, 3 &4, 4 & 5, and 5 & 1 are expected to alternate between all repetitions of the events. This is shown in Fig. 1(b), where each pair of alternating tasks is connected by an edge. One possible task execution sequence that obeys these recurrence constraints – and no other – is shown in Fig. 1(a). Later in the paper, we will introduce an orientation of such graphs that we call a *semi-transitive orientation*; such an orientation for our example is shown in Fig. 1(c).

Execution sequences of recurring tasks can be viewed as words over an alphabet V, where V is the set of tasks.

Related work. The notion of directed word-representable graphs was introduced in [13] to obtain asymptotic bounds on the free spectrum of the widely-studied *Perkins semigroup*, which has played central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. In [12], numerous properties of word-representable graphs were