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Further hardness results on rainbow and strong rainbow connectivity

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ABSTRACT

A path in an edge-colored graph is *rainbow* if no two edges of it are colored the same. The graph is said to be *rainbow connected* if there is a rainbow path between every pair of vertices. If there is a rainbow shortest path between every pair of vertices, the graph is *strong rainbow connected*. We consider the complexity of the problem of deciding if a given edge-colored graph is rainbow or strong rainbow connected. These problems are called Rainbow connectivity and Strong rainbow connectivity, respectively. We prove both problems remain NP-complete on interval outerplanar graphs and k-regular graphs for $k \geq 3$. Previously, no graph class was known where the complexity of the two problems would differ. We show that for block graphs, which form a subclass of chordal graphs, Rainbow connectivity is NP-complete while Strong rainbow connectivity is in P. We conclude by considering some tractable special cases, and show for instance that both problems are in XP when parameterized by tree-depth.

1. Introduction

Let G be an edge-colored undirected graph that is simple and finite. A path in G is *rainbow* if no two edges of it are colored the same. The graph G is *rainbow connected* if there is a rainbow path between every pair of vertices. If there is a rainbow shortest path between every pair of vertices, G is *strong rainbow connected*. Clearly, a strong rainbow connected graph is also rainbow connected. The minimum number of colors needed to make G rainbow connected is known as the *rainbow connection number* and is denoted by $\mathrm{rc}(G)$. Likewise, the minimum number of colors needed to make G strong rainbow connected is known as the *strong rainbow connection number* and is denoted by $\mathrm{src}(G)$. The concept of rainbow connectivity was introduced by Chartrand et al. [4] in 2008, and it has applications in data transfer and networking. The *diameter* of a graph, denoted by $\mathrm{diam}(G)$, is the largest distance between two vertices of G. Clearly, $\mathrm{diam}(G)$ is a lower bound for $\mathrm{rc}(G)$. On the other hand, a trivial upper bound for $\mathrm{rc}(G)$ is m, where m is the number of edges in G. Finally, because each strong rainbow connected graph is also rainbow connected, we have that $\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G) \leq m$. For less trivial bounds and more, we refer the reader to the books [5,18], or the recent survey [17].

A similar concept was introduced for vertex-colored graphs by Krivelevich and Yuster [14]. A vertex-colored graph H is *rainbow vertex-connected* if every pair of vertices is connected by a path whose internal vertices have distinct colors. The minimum number of colors needed to make H rainbow vertex-connected is known as the *rainbow vertex-connection number* and is denoted by $\mathrm{rvc}(H)$. Li et al. [16] investigated the *strong rainbow vertex-connection number* as a natural variant. A vertex-colored graph is *strong rainbow vertex-connected* if every pair of vertices is connected by a shortest path whose internal vertices have distinct colors. The minimum number of colors needed to make H strong rainbow vertex-connected

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is known as the *strong rainbow vertex-connection number* and is denoted by $\mathrm{srvc}(H)$. For rainbow vertex-connection numbers or other rainbow connection numbers outside of our scope we refer the reader to [17].

Rainbow connectivity can be motivated by the following example from the domain of networking. Suppose we have a network of agents represented as a graph. Each vertex in the graph represents an agent, and an edge between two agents is a link. An agent in the network wishes to communicate with every other agent in the network by sending messages. A message sent from agent A to agent B is routed through other agents that act as intermediaries. This communication path uses links between agents, and each link uses a channel. For the message to get through, we require that each link on the communication path receives a distinct channel. Given a network of agents G, our objective is to ensure each pair of agents can establish a communication path, while also minimizing the number of channels needed. The minimum number of channels we need is exactly $\mathrm{rc}(G)$.

Chakraborty et al. [2] showed that it is NP-complete to decide if $\mathrm{rc}(G) \le k$ for $k = 2$. Ananth et al. [1] proved the problem remains hard for $k \ge 3$ as well. Chandran and Rajendraprasad [3] proved there is no polynomial time algorithm to rainbow color graphs with less than twice the optimum number of colors, unless P = NP. Computing the strong rainbow connection number is known to be hard as well. Chartrand et al. [4] proved $\mathrm{rc}(G) = 2$ if and only if $\mathrm{src}(G) = 2$, so deciding if $\mathrm{src}(G) \le k$ is NP-complete for $k = 2$. Ananth et al. [1] showed the problem remains NP-complete for $k \ge 3$ even when G is bipartite [1]. In the same paper, they also showed there is no polynomial time algorithm for approximating the strong rainbow connection number of an n-vertex graph within a factor of $n^{1/2-\epsilon}$, where $\epsilon > 0$ unless NP = ZPP.

Given that it is hard to compute both the rainbow and the strong rainbow connection number, it is natural to ask if it is easier to verify if a given edge-colored graph is rainbow or strong rainbow connected. In this paper, we are concerned with the complexity of the following two decision problems:

RAINBOW CONNECTIVITY
Instance: An undirected graph $G = (V, E)$, and an edge-coloring $\chi : E \to C$, where C is a set of colors
Question: Is G rainbow connected under χ?

STRONG RAINBOW CONNECTIVITY
Instance: An undirected graph $G = (V, E)$, and an edge-coloring $\chi : E \to C$, where C is a set of colors
Question: Is G strong rainbow connected under χ?

Out of these two problems, RAINBOW CONNECTIVITY has gained considerably more attention in the literature. Chakraborty et al. [2] observed the problem is easy when the number of colors $|C|$ is bounded from above by a constant. However, they proved that for an arbitrary coloring, the problem is NP-complete. Building on their result, Li et al. [15] proved RAINBOW CONNECTIVITY remains NP-complete for bipartite graphs. Furthermore, the problem is NP-complete even for bipartite planar graphs as shown by Huang et al. [11]. Recently, Uchizawa et al. [23] complemented these results by showing RAINBOW CONNECTIVITY is NP-complete for outerplanar graphs, and even for series–parallel graphs. In the same paper, the authors also gave some positive results. Namely, they showed the problem is in P for cactus graphs, which form a subclass of outerplanar graphs. Furthermore, they settled the precise complexity of the problem from a viewpoint of graph diameter by showing the problem is in P for graphs of diameter 1, but NP-complete already for graphs of diameter greater than or equal to 2. To the best of our knowledge, Uchizawa et al. [23] were the only ones to consider STRONG RAINBOW CONNECTIVITY. They showed the problem is in P for cactus graphs, but NP-complete for outerplanar graphs. We shortly mention similar hardness results are known for deciding if a given vertex-colored is rainbow vertex-connected (see e.g. [6,15,23]).

A *fixed-parameter algorithm* (FPT) solves a problem with an input instance of size n and a parameter k in $f(k) \cdot n^{O(1)}$ time for some computable function f depending solely on k. That is, for every fixed parameter value it yields a solution in polynomial time and the degree of the polynomial is independent from k. Uchizawa et al. [23] gave FPT algorithms for both problems on general graphs when parameterized by the number of colors $k = |C|$. These algorithms run in $O(k2^k mn)$ time and $O(k2^k n)$ space, where n and m are the number of vertices and edges in the input graph, respectively. These algorithms imply both RAINBOW CONNECTIVITY and STRONG RAINBOW CONNECTIVITY are solvable in polynomial time for any n-vertex graph if $|C| = O(\log n)$.

In this paper, we prove both RAINBOW CONNECTIVITY and STRONG RAINBOW CONNECTIVITY remain NP-complete for interval outerplanar graphs. We then consider the class of block graphs, which form a subclass of chordal graphs. Interestingly, for block graphs RAINBOW CONNECTIVITY is NP-complete, while STRONG RAINBOW CONNECTIVITY is in P. To the best of our knowledge, this is the first graph class known for which the complexity of these two problems differ. Both problems are easy on 2-regular graphs. However, we show that both problems become NP-complete on cubic graphs, and further generalize this for k-regular graphs, where $k > 3$. This completely settles the complexity of both problems from the viewpoint of regularity.

2. Preliminaries

All graphs in this paper are simple, finite, and undirected. We begin by defining the graph classes we consider in this work. For graph theoretic concepts not defined here, we refer the reader to [8]. For an integer n, we write $[n] = \{1, 2, \ldots, n\}$.