Discrete Applied Mathematics 201 (2016) 201–212

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On orthogonal ray trees

Irina Mustaţă [a], Kousuke Nishikawa [b], Asahi Takaoka [b,*], Satoshi Tayu [b], Shuichi Ueno [b]

[a] *Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany*

[b] *Department of Communications and Computer Engineering, Tokyo Institute of Technology, Tokyo 152–8550-S3-57, Japan*

ARTICLE INFO

Article history:
Received 27 February 2015
Received in revised form 25 July 2015
Accepted 31 July 2015
Available online 24 August 2015

Keywords:
Edge-asteroids
Forbidden minors
Intersection graphs
Orthogonal ray graphs
Orthogonal ray trees

ABSTRACT

An orthogonal ray graph is an intersection graph of horizontal rays (closed half-lines) and vertical rays in the plane, which is introduced in connection with the defect-tolerant design of nano-circuits. An orthogonal ray graph is a 3-directional orthogonal ray graph if every vertical ray has the same direction. A 3-directional orthogonal ray graph is a 2-directional orthogonal ray graph if every horizontal ray has the same direction. The characterizations and the complexity of the recognition problem have been open for orthogonal ray graphs and 3-directional orthogonal ray graphs, while various characterizations with a quadratic-time recognition algorithm have been known for 2-directional orthogonal ray graphs. In this paper, we show several characterizations with a linear-time recognition algorithm for orthogonal ray trees by using the 2-directional orthogonal ray trees. We also show that a tree is a 3-directional orthogonal ray graph if and only if it is a 2-directional orthogonal ray graph. Moreover, we show some necessary conditions for orthogonal ray graphs and 3-directional orthogonal ray graphs.

1. Introduction

A graph G is called an *intersection graph* if there exists a set of objects such that each vertex corresponds to an object and two vertices are adjacent if and only if the corresponding objects intersect. Such a set of objects is called a *representation* of G. Intersection graphs of geometric objects have been extensively investigated, since the representations allow us to design efficient algorithms. The intersection graphs of geometric objects have many applications in various areas including integrated circuits, scheduling, and bioinformatics. See [3,11,18,28] for survey.

Segment graphs are the intersection graphs of straight-line segments in the plane, and one of the most natural and well-studied classes of the intersection graphs [5,16]. The recognition problem for segment graphs is known to be NP-hard [17]. A segment graph is called a *grid intersection graph* [1,12] if the lines are restricted to being parallel to the x- and y-axes (horizontal and vertical) such that no two parallel segments intersect. The recognition problem for grid intersection graphs is also known to be NP-complete [15]. A grid intersection graph is called a *unit grid intersection graph* [21] if every line segments have the same (unit) length. Recently, it has been shown in [20] that the recognition problem for unit grid intersection graphs is NP-complete.

Besides the segment graphs, the intersection graphs of rays (closed half-lines) in the plane have been considered [4,14,25]. We focus on the case where every rays are parallel to the x- and y-axes. Such intersection graphs are called orthogonal ray

* Corresponding author. Tel.: +81 3 5734 2565; fax: +81 3 5734 2902.
E-mail addresses: irina.mh.mustata@gmail.com (I. Mustaţă), asahi@eda.ce.titech.ac.jp (A. Takaoka), tayu@eda.ce.titech.ac.jp (S. Tayu), ueno@eda.ce.titech.ac.jp (S. Ueno).

http://dx.doi.org/10.1016/j.dam.2015.07.034
0166-218X/

graphs [25]. Formally, a bipartite graph G with bipartition (U, V) is called an *orthogonal ray graph* (*ORG* for short) if there exist a set of disjoint horizontal rays R_u, $u \in U$, in the xy-plane, and a set of disjoint vertical rays R_v, $v \in V$, such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. A set $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$ is called an *orthogonal ray representation* of G. The ORGs are introduced in connection with the defect-tolerant design of nano-circuits [24]. An ORG G with bipartition (U, V) is called a *3-directional orthogonal ray graph* (*3-DORG* for short) if G has an orthogonal ray representation $\mathcal{R}(G)$ such that every vertical ray $R_v \in \mathcal{R}(G)$, $v \in V$, has the same direction. An ORG G with bipartition (U, V) is called a *2-directional orthogonal ray graph* (*2-DORG* for short) if G has an orthogonal ray representation $\mathcal{R}(G)$ such that every horizontal ray $R_u \in \mathcal{R}(G)$, $u \in U$, has the same direction and every vertical ray $R_v \in \mathcal{R}(G)$, $v \in V$, has the same direction.

Among the graph classes above, the following relationship has been known [25]: {2-Directional Orthogonal Ray Graphs} $\subset$ {Orthogonal Ray Graphs} $\subset$ {Unit Grid Intersection Graphs} $\subset$ {Grid Intersection Graphs} $\subset$ {Bipartite Graphs}, where $X \subset Y$ indicates a set X is a proper subset of Y.

The 2-DORGs have been well investigated [8,22–27,30,31], and various characterizations have been known [25,30]. One of the characterizations is that 2-DORGs are the complements of circular-arc graphs with clique cover number 2, which is a well-studied class of graphs [9,13,29,32]. Based on the characterization, 2-DORGs can be recognized in $O(n^2)$ time, where n is the number of vertices in a graph. The 2-DORGs also has a forbidden graph characterization such that a bipartite graph is a 2-DORG if and only if it contains no induced cycle of length at least 6 or edge-asteroids [9,25].

On the other hand, the characterizations and the complexity of the recognition problem have been open for ORGs and 3-DORGs. As the first step to understand ORGs and 3-DORGs, it is natural to study the case of trees. A tree is called an *orthogonal ray tree* (*ORT* for short) if it is an orthogonal ray graph. An ORT is called a *3-directional orthogonal ray tree* (*3-DORT* for short) if it is a 3-DORG, and called a *2-directional orthogonal ray tree* (*2-DORT* for short) if it is a 2-DORG. The 2-DORTs have been investigated, and several characterizations with a linear-time recognition algorithm are known [24,25]. We have also known that any tree is a unit grid intersection graph [21].

The purpose of the paper is to show several characterizations with a linear-time recognition algorithm for ORTs and 3-DORTs by using the characterizations of 2-DORTs. We also show some necessary conditions for ORGs and 3-DORGs.

We show in Section 2 some characterizations for 2-DORGs and 2-DORTs used in this paper. In Section 3, we introduce a new forbidden structure, an asteroidal quintuple of edges (A5E for short), and show that any ORG contains no A5Es, which is also a sufficient condition for ORTs as shown in Section 4. We also show in Section 4 that any ORT is a graph obtained from two 2-DORTs by identifying a vertex in one 2-DORT with a vertex in the other. Moreover, we show a forbidden minor characterization with a linear-time recognition algorithm for ORTs. In Section 5, we show that any 3-DORG contains no edge-asteroids, and hence, a tree is a 3-DORT if and only if it is a 2-DORT.

The characterizations and the complexity of the recognition problem for ORGs and 3-DORGs still remain interesting open questions.

2. Two-directional orthogonal ray graphs

We show in this section some preliminaries and several characterizations for 2-DORGs and 2-DORTs used in this paper. See [24,25] for more information.

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E(G)\}$, and the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For an edge $e = (u, v)$ of G, we use $N_G[e]$ to denote the set of vertices adjacent to u or v, that is, $N_G[e] = N_G[u] \cup N_G[v]$. If no confusion arises, we will omit the index G.

A bipartite graph is called a *chordal bipartite graph* if it contains no induced cycles of length at least 6.

Let P be a path of length k with $V(P) = \{v_0, v_1, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$, where $e_i = (v_{i-1}, v_i)$, $1 \le i \le k$. We refer to P as a path from e_1 to e_k. A set of edges $\{e_0, e_1, \ldots, e_{2k}\} \subseteq E(G)$, $k \ge 1$, of a graph G is called an *edge-asteroid* of size $2k+1$ if for any i, $0 \le i \le 2k$, there exists a path from e_i to e_{i+1} that contains no vertices in $N[e_{i+k+1}]$ (subscripts are modulo $2k+1$). See Fig. 1 for examples of edge-asteroids. Edge-asteroids are introduced in [9], and 2-DORGs can be characterized as follows.

Theorem A ([9,25]). *A bipartite graph is a 2-DORG if and only if it is a chordal bipartite graph and contains no edge-asteroids.* □

The graph obtained from a complete bipartite graph $K_{1,3}$ (which is also known as a *claw*) by replacing each edge with a path of length 3 is called a *3-claw*. The 3-claw contains an edge-asteroid as shown in Fig. 1(c). A path P in a tree T is called a *spine* of T if every vertex of T is within distance 2 from a vertex on P. It has been known that 2-DORTs can be characterized as follows.

Theorem B ([24,25]). *The following statements are equivalent for a tree T:*

(i) *T is a 2-DORT;*
(ii) *T contains no 3-claw as a subtree;*
(iii) *T has a spine.* □

Theorem B implies a linear-time recognition algorithm for 2-DORTs [25], since it suffices to verify whether a longest path in a given tree is a spine, and a longest path in a tree can be obtained in linear time [7].