Discrete Applied Mathematics 201 (2016) 222–227

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On r-dynamic chromatic number of graphs

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ARTICLE INFO

Article history:
Received 30 June 2014
Received in revised form 9 March 2015
Accepted 20 July 2015
Available online 10 August 2015

Keywords:
Chromatic number
r-dynamic chromatic number
Forest transversal

ABSTRACT

An r-dynamic k-coloring of a graph G is a proper vertex k-coloring of G such that the neighbors of any vertex v receive at least $\min\{r, \deg(v)\}$ different colors. The r-dynamic chromatic number of G, $\chi_r(G)$, is defined as the smallest k such that G admits an r-dynamic k-coloring. In this paper, first we introduce an upper bound for $\chi_r(G)$ in terms of r, chromatic number, maximum degree and minimum degree. Then, motivated by a conjecture of Montgomery (2001) stating that for a d-regular graph G, $\chi_2(G) - \chi(G) \leq 2$, we prove two upper bounds for $\chi_2 - \chi$ on regular graphs. Our first upper bound $\lceil 5.437 \log d + 2.721 \rceil$ improves a result of Alishahi (2011). Also, our second upper bound shows that Montgomery's conjecture is implied by the existence of a $\chi(G)$-coloring for any regular graph G, such that any two vertices whose neighbors are unicolored in this coloring, have no common neighbor.

1. Introduction

We consider finite, simple, undirected graphs $G = (V(G), E(G))$, on the vertex set $V(G)$ and the edge set $E(G)$. For a vertex v of G, the set of neighbors of v is denoted by $N_G(v)$ (or $N(v)$ if there is no confusion) and the degree of v, i.e., $|N_G(v)|$, is denoted by $\deg(v)$. The minimum and maximum of $|N_G(v)|$ over $v \in V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Also, for $B \subseteq V(G)$, the set of neighbors of B in G is denoted by $N_G(B)$ (or $N(B)$ if there is no confusion), and the subgraph of G induced on B is denoted by $G[B]$. The second power of G, denoted by G^2, is a graph on the vertex set $V(G)$, in which two distinct vertices are adjacent if and only if their distance in G is at most 2. In the sequel, log stands for the natural logarithm function and e denotes its base.

Let r be a positive integer. An r-dynamic k-coloring of a graph G is a proper vertex k-coloring such that every vertex v receives at least $\min\{r, \deg(v)\}$ colors in its neighbors. The minimum k for which a graph G admits an r-dynamic k-coloring is called the r-dynamic chromatic number of G, and is denoted by $\chi_r(G)$. The r-dynamic chromatic number was first introduced by B. Montgomery [10] and the case $r = 2$ is usually called the dynamic chromatic number (e.g. [1–3,5,7,8]).

Our main motivation for this study and the subsequent results is the following conjecture of B. Montgomery.

Conjecture 1 (*[10]*). *For every regular graph G, we have $\chi_2(G) - \chi(G) \leq 2$.*

In what follows we first concentrate on upper bounds of $\chi_r(G)$ in general. In this regard we prove a basic result that will be used later, stating that $r\chi(G)$ is an upper bound for $\chi_r(G)$, when $e((\delta\Delta - \delta + 1)(r - 1) + 1)(1 - 1/r)^{\delta} \leq 1$. It should be noted that Jahanbekam et al. [6] independently (and using the same method) proved this result for regular graphs. Moreover, using a greedy coloring procedure, they proved that $\chi_r(G) \leq r\Delta(G) + 1$.

Next, applying the probabilistic method, we establish our main result in the general case as an upper bound for $\chi_r(G)$ in terms of $\chi(G)$, $\Delta(G)$, $\delta(G)$, and r, as follows.

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http://dx.doi.org/10.1016/j.dam.2015.07.019

Theorem 1. *Let r be a positive integer with $2 \le r \le \delta/\log(2er(\Delta^2+1))$. Then we have,*

$$\chi_r(G) \le \chi(G) + (r-1)\left\lceil e\frac{\Delta}{\delta}\log(2er(\Delta^2+1))\right\rceil.$$

The second part of our contribution is obtained by a focus on the case $r = 2$. In this regard one should first note the following classical result for general graphs.

Theorem A (*[10]*). *For any graph G, we have $\chi_2(G) \le \Delta(G) + 3$.*

It is also instructive to note that if $\Delta(G) \ge 3$, then $\chi_2(G) \le \Delta(G) + 1$ (see [8]).

On the other hand, for relations between $\chi_2(G)$ and $\chi(G)$, and in particular on upper bounds for $\chi_2(G) - \chi(G)$, we know that the difference can be arbitrarily large on the whole class of simple graphs [10], while M. Alishahi has proved that if $\chi(G) \ge 4$ then $\chi_2(G) \le \chi(G) + \gamma(G)$, where $\gamma(G)$ is the domination number of G (see [3]).

Note that by Theorem 1 the role of $\frac{\Delta(G)}{\delta(G)}$ is settled on the upper bound. Hence, as a generalization of Conjecture 1, one may ask whether there exists a constant c_0 such that the above mentioned chromatic difference is controlled by $c_0(\Delta(G)/\delta(G))$ on the whole class of simple graphs.

In this regard, the upper bound of Theorem 1 for $r = 2$ and $\delta \ge 2\log(4e(\Delta^2+1))$ shows that

$$\chi_2(G) - \chi(G) \le \lceil e(\Delta/\delta)\log(4e(\Delta^2+1))\rceil.$$

Our next result indicates that for $r = 2$ this upper bound can be improved by removing the factor 4.

Given a coloring c of a graph G, a vertex v of G is said to be *bad* if $\deg(v) \ge 2$ and only one of the colors of c appears on the neighbors of v. Let B_c be the set of all bad vertices with respect to the coloring c of G. Note that every coloring c of G with $B_c = \varnothing$ is a dynamic coloring. Moreover, in [3], it is shown that if $\chi(G) \ge 4$, then for every $k \ge \chi(G)$, there is a proper k-coloring c such that the set of bad vertices, B_c, is an independent set.

Considering the above mentioned fact, and an efficient estimation of the probability of some events, we can prove the following theorem.

Theorem 2. *For any graph G we have, $\chi_2(G) \le \chi(G) + \lceil e\frac{\Delta}{\delta}\log(e(\Delta^2+1))\rceil$.*

Motivated by Conjecture 1 we further specialize to the case of $r = 2$ and regular graphs for which we know that,

Theorem B (*[2]*). *If G is a d-regular graph, then $\chi_2(G) \le \chi(G) + 14.06\log d + 1$.*

Also, M. Alishahi in [3] proves that for any d-regular graph G with no induced C_4, we have $\chi_2(G) \le \chi(G) + 2\lceil 4\log d + 1\rceil$. Note that the following corollary of Theorem 2 shows that $\chi_2(G) \le \chi(G) + \lceil 5.437\log d + 2.721\rceil$ when $d \ge 3$, which is an improvement of the above mentioned result.

Corollary 1. *If G is a d-regular graph, then $\chi_2(G) \le \chi(G) + \lceil e\log(e(d^2+1))\rceil$.*

As an epilogue to our second result recall that for a regular graph G, $\chi_2(G) \le \chi(G) + 2\log_2\alpha(G) + 3$ in which $\alpha(G)$ is the independence number of G (see [5]).

It is shown in [3] that for any d-regular graph G with $\chi(G) \ge 4$, we have $\chi_2(G) \le \chi(G) + \alpha(G^2)$. In our next result we generalize the above setup to a more flexible upper bound for $\chi_2(G) - \chi(G)$ in terms of the chromatic number of a graph strongly dependent on G^2, where the proof uses forest transversals in G (for more on independent transversal and its generalization see [11]).

Theorem 3. *Let G be a d-regular graph and c be a k-coloring of G. If B_c is the set of all bad vertices for coloring c, then*

$$\chi_2(G) \le \min\{k + 2\chi(G^2[B_c] \setminus E(G)) : c \text{ is a } k\text{-coloring of } G\}.$$

First, note that as a direct consequence of Theorem 3 we have

$$\chi_2(G) \le k + 2\chi(G^2[B_c] \setminus E(G))$$

for any d-regular graph G and any k-coloring c of G. On the other hand, if one can color a d-regular graph G with $\chi(G)$ colors such that any two bad vertices of G in this coloring have no common neighbor, then

$$\chi_2(G) - \chi(G) \le 2.$$

Clearly, this provides a possible approach to prove Conjecture 1.