Discrete Applied Mathematics 201 (2016) 228–247

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Further result on acyclic chromatic index of planar graphs

Tao Wang [a,b,*], Yaqiong Zhang [b]

[a] Institute of Applied Mathematics, Henan University, Kaifeng, 475004, PR China

[b] College of Mathematics and Information Science, Henan University, Kaifeng, 475004, PR China

ARTICLE INFO

Article history:
Received 5 May 2014
Received in revised form 1 July 2015
Accepted 16 July 2015
Available online 24 August 2015

Keywords:
Acyclic edge coloring
Acyclic chromatic index
κ-deletion-minimal graph
κ-minimal graph
Acyclic edge coloring conjecture

ABSTRACT

An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G. It was conjectured that $\chi'_a(G) \leq \Delta(G)+2$ for any simple graph G with maximum degree $\Delta(G)$. In this paper, we prove that every planar graph G admits an acyclic edge coloring with $\Delta(G) + 6$ colors.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G. It is obvious that $\chi'_a(G) \geq \chi'(G) \geq \Delta(G)$. Fiamčík [5] stated the following conjecture in 1978, which is well known as Acyclic Edge Coloring Conjecture, and Alon et al. [2] restated it in 2001.

Conjecture 1. *For any graph G, $\chi'_a(G) \leq \Delta(G) + 2$.*

Alon et al. [1] proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph G by using probabilistic method. Molloy and Reed [11] improved it to $\chi'_a(G) \leq 16\Delta(G)$. Recently, Ndreca et al. [12] improved the upper bound to $\lceil 9.62(\Delta(G) - 1)\rceil$, and Esperet and Parreau [4] further improved it to $4\Delta(G) - 4$ by using the so-called entropy compression method. The best known general bound is $\lceil 3.74(\Delta(G) - 1)\rceil$ due to Giotis et al. [7]. Alon et al. [2] proved that there is a constant c such that $\chi'_a(G) \leq \Delta(G)+2$ for a graph G whenever the girth is at least $c\Delta \log \Delta$.

Regarding general planar graph G, Fiedorowicz et al. [6] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$; Hou et al. [10] proved that $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 22\}$. Recently, Basavaraju et al. [3] showed that $\chi'_a(G) \leq \Delta(G) + 12$, and Guan et al. [8] improved it to $\chi'_a(G) \leq \Delta(G) + 10$, and Wang et al. [14] further improved it to $\chi'_a(G) \leq \Delta(G) + 7$.

In this paper, we improve the upper bound to $\Delta(G) + 6$ by the following theorem.

Theorem 1.1. *If G is a planar graph, then $\chi'_a(G) \leq \Delta(G) + 6$.*

* Corresponding author at: Institute of Applied Mathematics, Henan University, Kaifeng, 475004, PR China.
E-mail address: wangtao@henu.edu.cn (T. Wang).

http://dx.doi.org/10.1016/j.dam.2015.07.015
0166-218X/

2. Preliminary

Let $\mathbb{S}$ be a multiset and x be an element in $\mathbb{S}$. The *multiplicity* $\mathrm{mul}_{\mathbb{S}}(x)$ is the number of times x appears in $\mathbb{S}$. Let $\mathbb{S}$ and $\mathbb{T}$ be two multisets. The union of $\mathbb{S}$ and $\mathbb{T}$, denoted by $\mathbb{S} \uplus \mathbb{T}$, is a multiset with $\mathrm{mul}_{\mathbb{S}\uplus\mathbb{T}}(x) = \mathrm{mul}_{\mathbb{S}}(x) + \mathrm{mul}_{\mathbb{T}}(x)$. Throughout this paper, every coloring uses colors from $[\kappa] = \{1, 2, \dots, \kappa\}$.

We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices that are adjacent to v in G and $\deg_G(v)$ (or simple $\deg(v)$) to denote the degree of v in G. When G is a plane graph, we use $F(G)$ to denote its face set and $\deg_G(f)$ (or simple $\deg(f)$) to denote the degree of a face f in G. A k-, k^+-, k^--vertex (resp. face) is a vertex (resp. face) with degree k, at least k and at most k, respectively. A face $f = v_1v_2 \dots v_k$ is a $(\deg(v_1), \deg(v_2), \dots, \deg(v_k))$-face.

A graph G with maximum degree at most κ is *κ-deletion-minimal* if $\chi'_a(G) > \kappa$ and $\chi'_a(H) \leq \kappa$ for every proper subgraph H of G. A graph property $\mathscr{P}$ is *deletion-closed* if $\mathscr{P}$ is closed under taking subgraphs. Analogously, we can define another type of minimal graphs by taking minors. A graph G with maximum degree at most κ is *κ-minimal* if $\chi'_a(G) > \kappa$ and $\chi'_a(H) \leq \kappa$ for every proper minor H with $\Delta(H) \leq \Delta(G)$. Obviously, every proper subgraph of a κ-minimal graph admits an acyclic edge coloring with at most κ colors, and then every κ-minimal graph is also a κ-deletion-minimal graph and all the properties of κ-deletion-minimal graphs are also true for κ-minimal graphs.

Let G be a graph and H be a subgraph of G. An acyclic edge coloring of H is a *partial acyclic edge coloring* of G. Let $\mathcal{U}_\phi(v)$ denote the set of colors which are assigned to the edges incident with v with respect to ϕ. Let $C_\phi(v) = [\kappa] \setminus \mathcal{U}_\phi(v)$ and $C_\phi(uv) = [\kappa] \setminus (\mathcal{U}_\phi(u) \cup \mathcal{U}_\phi(v))$. Let $\Upsilon_\phi(uv) = \mathcal{U}_\phi(v) \setminus \{\phi(uv)\}$ and $W_\phi(uv) = \{\, u_i \mid uu_i \in E(G) \text{ and } \phi(uu_i) \in \Upsilon_\phi(uv) \,\}$. Notice that $W_\phi(uv)$ may be not same with $W_\phi(vu)$. For simplicity, we will omit the subscripts if no confusion can arise.

An (α, β)-maximal dichromatic path with respect to ϕ is a maximal path whose edges are colored by α and β alternately. An (α, β, u, v)-critical path with respect to ϕ is an (α, β)-maximal dichromatic path which starts at u with color α and ends at v with color α. An (α, β, u, v)-alternating path with respect to ϕ is an (α, β)-dichromatic path starting at u with color α and ending at v with color β.

Let ϕ be a partial acyclic edge coloring of G. A color α is *candidate* for an edge e in G with respect to a partial edge coloring of G if none of the adjacent edges of e is colored with α. A candidate color α is *valid* for an edge e if assigning the color α to e does not result in any dichromatic cycle in G.

Fact 1 (*Basavaraju et al. [3]*)**.** Given a partial acyclic edge coloring of G and two colors α, β, there exists at most one (α, β)-maximal dichromatic path containing a particular vertex v. $\square$

Fact 2 (*Basavaraju et al. [3]*)**.** Let G be a κ-deletion-minimal graph and uv be an edge of G. If ϕ is an acyclic edge coloring of $G - uv$, then no candidate color for uv is valid. Furthermore, if $\mathcal{U}(u) \cap \mathcal{U}(v) = \emptyset$, then $\deg(u) + \deg(v) = \kappa + 2$; if $|\mathcal{U}(u) \cap \mathcal{U}(v)| = s$, then $\deg(u) + \deg(v) + \sum_{w \in W(uv)} \deg(w) \geq \kappa + 2s + 2$. $\square$

We remind the readers that we will use these two facts frequently, so please keep these in mind and we will not refer it at every time.

3. Structural lemmas

Wang and Zhang [13] presented many structural results on κ-deletion-minimal graphs and κ-minimal graphs. In this section, we give more structural lemmas in order to prove our main result.

Lemma 1. *If G is a κ-deletion-minimal graph, then G is 2-connected and $\delta(G) \geq 2$.*

3.1. Local structure on the 2- or 3-vertices

Lemma 2 (*Wang and Zhang [13]*)**.** *Let G be a κ-minimal graph with $\kappa \geq \Delta(G) + 1$. If v_0 is a 2-vertex of G, then v_0 is contained in a triangle.*

Lemma 3 (*Wang and Zhang [13]*)**.** *Let G be a κ-deletion-minimal graph. If v is adjacent to a 2-vertex v_0 and $N_G(v_0) = \{w, v\}$, then v is adjacent to at least $\kappa - \deg(w) + 1$ vertices with degree at least $\kappa - \deg(v) + 2$. Moreover,*

(A) *if $\kappa \geq \deg(v) + 1$ and $wv \in E(G)$, then v is adjacent to at least $\kappa - \deg(w) + 2$ vertices with degree at least $\kappa - \deg(v) + 2$, and $\deg(v) \geq \kappa - \deg(w) + 3$;*

(B) *if $\kappa \geq \Delta(G) + 2$ and v is adjacent to precisely $\kappa - \Delta(G) + 1$ vertices with degree at least $\kappa - \Delta(G) + 2$, then v is adjacent to at most $\deg(v) + \Delta(G) - \kappa - 3$ vertices with degree two and $\deg(v) \geq \kappa - \Delta(G) + 4$.*

Lemma 4 (*Wang and Zhang [13]*)**.** *Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$. If v_0 is a 2-vertex, then every neighbor of v_0 has degree at least $\kappa - \Delta(G) + 4$.*