Discrete Applied Mathematics 200 (2016) 23–42

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Algorithmic aspects of switch cographs

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ARTICLE INFO

Article history:
Received 4 October 2013
Received in revised form 4 June 2015
Accepted 7 July 2015
Available online 6 August 2015

Keywords:
Graph algorithms and decomposition
2-structures
Modular decomposition
Involution modular decomposition
Switch cographs

ABSTRACT

This paper introduces the notion of involution module, the first generalization of the modular decomposition of 2-structures which has a unique linear-sized decomposition tree. We derive an $\mathcal{O}(n^2)$ decomposition algorithm and we take advantage of the involution modular decomposition tree to obtain several algorithmic results. It is well-known that cographs (or equivalently P_4-free graphs) are the graphs that are totally decomposable w.r.t modular decomposition. In a similar way, we introduce the class of switch cographs, as the class of graphs that are totally decomposable w.r.t involution modular decomposition. This class generalizes the class of cographs and appears to be exactly the class of (Bull, Gem, Co-Gem, C_5)-free graphs. We use our new decomposition tool to design three practical algorithms for the maximum cut, vertex cover and vertex separator problems. The complexity of these problems was still unknown for this class of graphs. This paper also improves the complexity of the maximum clique, the maximum independent set, the chromatic number and the maximum clique cover problems by giving efficient algorithms, thanks to the decomposition tree. Eventually, we show that this class of graphs has clique-width at most 4 and that a clique-width expression can be computed in linear time.

0. Introduction

Modular decomposition has arisen in different contexts as a very natural operation on many discrete structures such as graphs, directed graphs, 2-structures, automata, boolean functions, hypergraphs, or matroids. In graph theory, the study of modular decomposition as a graph decomposition technique was first introduced by Gallai [18]. This notion has led to state several important properties of both structural and algorithmic flavor. Many graph classes such as cographs, P_4-sparse graphs or P_4-tidy graphs are characterized by the properties of their modular decomposition (see for example [3]).

Also, several classical graph problems (NP-complete in the general case) can be solved in polynomial time when restricted to classes of graphs that are "*decomposable enough*". For example, [8,26] designed efficient algorithms for the class of cographs which rely on the modular decomposition tree of the cographs.

We start from a generalization of modular decomposition, namely the umodular decomposition defined in [12]. As far as we know, there is no generalization of modular decomposition that have a polynomial-sized tree representation in a more general context than graphs.

In this paper, we introduce the notion of *involution modules*, which is a generalization of modules but a restriction of umodules, and we show that the family of involution modules of any 2-structure has very strong properties. These properties are similar to the properties of modules, and lead us to derive in $\mathcal{O}(n^2)$ time a unique linear-sized decomposition tree for any 2-structure. To this aim we use a very interesting switch operator that generalizes to 2-structures the well-known Seidel Switch introduced by [27] and widely studied by [20,21,24,25].

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http://dx.doi.org/10.1016/j.dam.2015.07.008

Then we focus our study on the particular case of 2-structures with two colors, namely undirected graphs which are more concerned by the algorithmic aspects than 2-structures. We consider the class of graphs totally decomposable with respect to the involution modular decomposition. We call this class the class of *switch cographs* and we show that switch cographs are exactly the graphs with no induced Gem, Co-Gem, C_5 nor Bull subgraphs. This graphs family is already known in the literature (see for example [21]) and generalizes the widely studied class of cographs. Like the modular decomposition for cographs, the involution modular decomposition provides crucial algorithmic properties for the class of switch cographs. Using our decomposition approach we give efficient and practical algorithms for the class of switch cographs to well-known graph problems (NP-complete in the general case), namely the maximum cut and the vertex separator problems. The complexity of these problems was still unknown for this class of graphs. Since the clique-width of the switch cographs is bounded, the complexity of several graph problems depended on the celebrated Courcelle's theorem. This theorem implies in particular that the maximum clique, the maximum independent set, the chromatic number the vertex cover and the minimum clique cover problems can be solved in polynomial time the class of switch cographs. Nevertheless, the theorem induces a huge constant factor in the big-O notation and cannot be considered of practical interest. We then show that the involution modular decomposition tree can be used in order to derive a clique-width expression in linear time leading to a linear-time complexity for these problems. Then, we give easily implementable algorithms which ensure the same optimal complexity. Finally, we conclude this paper by showing that this class of graphs is strictly included in the class of graphs with clique-width at most 4.

The paper is organized as follows, Section 1 recalls definitions and the general framework of modular decomposition, Section 2 introduces the notion of involution modules, studies its properties and presents the decomposition algorithm. Section 3 is devoted to the study of switch cographs and to the algorithms we designed thanks to the involution modular decomposition. Eventually, we discuss the noteworthy outcomes and open questions that follow from our work.

1. Definitions

We recall some definitions about generalizations of modular decomposition (as they are given in [12]). Let X be a finite set. We say that two subsets $A, B \subseteq X$ are *overlapping* if the sets $A \cap B, A \setminus B, B \setminus A$ are not empty. Finally, we say that two sets $A, B \subseteq X$ are *crossing* if they are overlapping and $X \neq A \cup B$. We denote $A \triangle B$ the set symmetric difference $(A \cup B) - (A \cap B)$.

Definition 1.1 (*[16] 2-structure.*)**.** A 2-structure G is a couple (X, E) where X is a finite set (the set of the *vertices*) and E is a function, $E : X^2 \to \mathbb{N}$.

We say that a 2-structure G is symmetric if for all $x, y \in X$, $E(x, y) = E(y, x)$. An edge over X is a pair (x, y), $x, y \in X$ and $x \neq y$ and let $E_2(X)$ denotes the set of all edges over X. Throughout this paper, we only consider symmetric 2-structures and we always omit the word "symmetric". For a given 2-structure $G = (X, E)$ we always identify (up to renumbering) the set $C = \{i \mid \exists u, v \text{ s.t } E(u, v) = i\}$ of colors actually used in the 2-structure with $\{1, \dots, c\}$. By $N_s^i(X')$ we denote the set $\{x \mid x \in X' \text{ and } E(s, x) = i\}$, basically the set of elements in X' that are connected to s with the color i.

The reader may remark that any undirected graph is basically a 2-structure with $c = 2$ colors. Let us recall below the usual notation of modular decomposition.

1.1. Homogeneous relation, modules and umodules

We now recall the notion of module for a 2-structure.

Definition 1.2 (*[15] Modules*)**.** Let $G = (X, E)$ be a 2-structure. A subset $M \subseteq X$ is a module of G if:
$\forall m, m' \in M, \forall i \in C, N_m^i(X \setminus M) = N_{m'}^i(X \setminus M)$.

We say that a module M is trivial if $|M| \leq 1$ or $M = X$. We now present the primary properties of modular decomposition. Throughout this section, we denote by 2^X the family of subsets of any finite set X.

Definition 1.3 (*Partitive Family*)**.** Let X be a set of elements. $\mathcal{F} \subseteq 2^X$ is a partitive family if X and $\emptyset \in \mathcal{F}$ and for any overlapping sets $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$, $A \cup B \in \mathcal{F}$, $A \setminus B \in \mathcal{F}$ and $A \triangle B \in \mathcal{F}$.

A member of $\mathcal{F}$ is *strong* if it overlaps no other member $\mathcal{F}$. It is proved in [6] that the family of modules of any graph (i.e. 2-structure with two colors) is a partitive family, and demonstrated the following theorem of particular importance:

Theorem 1.1 (*[6] Decomposition Theorem of Partitive Families*)**.** *If F is a partitive family, there exists a unique rooted undirected tree-representation of F, $\mathcal{T}(F)$, of size $\mathcal{O}(|X|)$. This tree representation is such that the internal nodes of $\mathcal{T}(F)$ can be labeled complete or prime such that:*

- *The leaves are exactly the elements of X.*
- *Let N be a node with k siblings $N_1, \dots, N_k$, and X_i the leaf-set of N_i (set of elements of leaves whose paths to N traverse N_i). Then*
  the leaf-set of N is a strong member of $\mathcal{F}$, and
  $k \geq 2$, and
  if N is a complete node, for any $I \subset \{1, \dots, k\}$ such that $1 \leq |I| \leq k$, $\bigcup_{i \in I} X_i \in \mathcal{F}$.
- *There are no more sets in $\mathcal{F}$ than the ones described above.*