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Classifying the clique-width of H-free bipartite graphs[☆]

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ABSTRACT

Let G be a bipartite graph, and let H be a bipartite graph with a fixed bipartition (B_H, W_H). We consider three different, natural ways of forbidding H as an induced subgraph in G. First, G is H-free if it does not contain H as an induced subgraph. Second, G is strongly H-free if no bipartition of G contains an induced copy of H in a way that respects the bipartition of H. Third, G is weakly H-free if G has at least one bipartition that does not contain an induced copy of H in a way that respects the bipartition of H. Lozin and Volz characterized all bipartite graphs H for which the class of strongly H-free bipartite graphs has bounded clique-width. We extend their result by giving complete classifications for the other two variants of H-freeness.

1. Introduction

The *clique-width* of a graph G is a well-known graph parameter that has been studied both in a structural and in an algorithmic context. It is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;
(ii) taking the disjoint union of two labelled graphs G_1 and G_2;
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$);
(iv) renaming label i to j.

We refer to the surveys of Gurski [19] and Kamiński, Lozin and Milanič [21] for an in-depth study of the properties of clique-width.

We say that a class of graphs has *bounded* clique-width if every graph from the class has clique-width at most c for some constant c. As many NP-hard graph problems can be solved in polynomial time on graph classes of bounded clique-width [13,22,27,28], it is natural to determine whether a certain graph class has bounded clique-width and to find new graph classes of bounded clique-width. In particular, many papers have determined the clique-width of graph classes characterized by one or more forbidden induced subgraphs [1–12,15,16,18,20,23–26].

In this paper we focus on classes of bipartite graphs characterized by a forbidden induced subgraph H. A graph G is H-free if it does not contain H as an induced subgraph. If G is bipartite, then when considering notions for H-freeness, we may assume without loss of generality that H is bipartite as well. For bipartite graphs, the situation is more subtle as one can

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(a) $(2P_1)^b$ (b) $(2P_1)^{\overline{b}}$ (c) $(P_1)^b + (P_1)^{\overline{b}}$

Fig. 1. The three pairwise non-isomorphic labellings of $2P_1$. The labellings b and $\overline{b}$ will be formally defined later.

define the notion of freeness with respect to a fixed ordered bipartition (B_H, W_H) of H. This leads to two other notions (see also Section 2 for formal definitions). We say that a bipartite graph G is strongly H-free if no bipartition of G contains an induced copy of H in a way that respects the bipartition of H. Strongly H-free graphs have been studied with respect to their clique-width, although under less explicit terminology (see e.g. [21,24,25]). In particular, Lozin and Volz [25] completely determined those bipartite graphs H, for which the class of strongly H-free graphs has bounded clique-width (we give an exact statement of their result in Section 3). If G has at least one bipartition that does not contain an induced copy of H in a way that respects the bipartition of H, then G is said to be weakly H-free. As we shall see, any H-free graph is strongly H-free, and any strongly H-free graph is weakly H-free, whereas the two reverse statements do not always hold. Moreover, as far as we are aware, the notion of being weakly H-free has not been studied with respect to the clique-width of bipartite graphs.

Our Results: We completely classify the classes of H-free bipartite and weakly H-free bipartite graphs of bounded clique-width. In this way, we have identified a number of new graph classes of bounded clique-width. Before stating our classification results precisely in Section 3, we first give some terminology and examples in Section 2. In Section 4 we give the proofs of our results.

2. Preliminaries

We first give some terminology on general graphs and notation to denote various well-known graphs. In Section 2.1 we introduce labelled bipartite graphs. We illustrate the definitions of H-freeness, strong H-freeness and weak H-freeness of bipartite graphs with some examples. As we will explain, these examples also make clear that all three notions are different from each other.

General graphs: Let G and H be graphs. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. A bijection $f : V_G \to V_H$ is called a *(graph) isomorphism* when $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. If such a bijection exists, then G and H are *isomorphic*. Let $\{H_1, \dots, H_p\}$ be a set of graphs. A graph G is *$(H_1, \dots, H_p)$-free* if no H_i is an induced subgraph of G. If $p = 1$, we may write H_1-free instead of (H_1)-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote the disjoint union of r vertex-disjoint copies of G by rG.

Special Graphs: For $r \geq 1$, the graphs C_r, K_r, P_r denote the cycle, complete graph and path on r vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. If $r = 3$, the graph $K_{1,r}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ denote the tree that has only one vertex x of degree 3 and that has exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{h,i,j}$ is said to be a *subdivided claw*. A graph G is a *linear forest* if every connected component of G is a path.

2.1. Labelled bipartite graphs

A graph G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. Let H be a bipartite graph. We say that H is a *labelled* bipartite graph if we are also given a *black-and-white labelling* ℓ, which is a labelling that assigns either the colour "black" or the colour "white" to each vertex of H in such a way that the two resulting monochromatic colour classes B_H^ℓ and W_H^ℓ form a *bipartition* of V_H into two (possibly empty) independent sets. From now on we denote a graph H with such a labelling ℓ by $H^\ell = (B_H^\ell, W_H^\ell, E_H)$. Here the pair (B_H^ℓ, W_H^ℓ) is *ordered*, that is, $(B_H^\ell, W_H^\ell, E_H)$ and $(W_H^\ell, B_H^\ell, E_H)$ are different labelled bipartite graphs.

We say that two labelled bipartite graphs H_1^ℓ and $H_2^{\ell^*}$ are *isomorphic* if the (unlabelled) graphs H_1 and H_2 are isomorphic, and if in addition there exists an isomorphism $f : V_{H_1} \to V_{H_2}$ such that for all $u \in V_{H_1}$, $u \in W_{H_1}^\ell$ if and only if $f(u) \in W_{H_2}^{\ell^*}$. Moreover, if $H_1 = H_2$, then ℓ and ℓ^* are said to be *isomorphic* labellings. For example, the bipartite graphs $(\{u, v\}, \emptyset)$ and $(\{x, y\}, \emptyset)$ are isomorphic, and the labelled bipartite graph $(\{u, v\}, \emptyset, \emptyset)$ is isomorphic to the labelled bipartite graph $(\{x, y\}, \emptyset, \emptyset)$. However, $(\{x, y\}, \emptyset, \emptyset)$ is neither isomorphic to $(\emptyset, \{x, y\}, \emptyset)$ nor to $(\{x\}, \{y\}, \emptyset)$ (also see Fig. 1).

We write $H_1^\ell \subseteq_{li} H_2^{\ell^*}$ if $H_1 \subseteq_i H_2$, $B_{H_1}^\ell \subseteq B_{H_2}^{\ell^*}$ and $W_{H_1}^\ell \subseteq W_{H_2}^{\ell^*}$. In this case we say that H_1^ℓ is a *labelled* induced subgraph of $H_2^{\ell^*}$. Note that the two labelled bipartite graphs $H_1^{\ell_1}$ and $H_2^{\ell_2}$ are isomorphic if and only if $H_1^{\ell_1}$ is a labelled induced subgraph of $H_2^{\ell_2}$, and vice versa.

Let G be an (unlabelled) bipartite graph, and let H^ℓ be a labelled bipartite graph. The graph G is *strongly H^ℓ-free* if for every labelling ℓ^* of G, G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. The graph G is *weakly H^ℓ-free* if there is a labelling ℓ^* of G such that G^{ℓ^*} does not contain H^ℓ as a labelled induced subgraph. Note that these two notions of freeness