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A measure for a balanced workload and its extremal values

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ABSTRACT

In order to measure the extent to which the distribution of workload between actors in the network can be equalized, a degree-weighted measure for a balanced workload based on betweenness centrality is introduced. The goal of this study is to determine the extremal values of the introduced measure, as well as the graph structures where the extremal values are attained. Several real world networks were used for evaluation of the new invariant. The obtained results are used for statistical comparison with standard measures of centrality to demonstrate validity of the introduced measure.

1. Introduction

In this work we are interested in the large scale systems which can be shaped as a network consisting of entities and interactions between them and which are organized in such a way that each entity interacts only with its immediate neighbours. For those systems the network size is not crucial, since the entities only need to know the local information in the network.

An representative example of such system is multi-agent system with its entities, i.e. agents which could equally well be robots, sensors, processes or persons. Considering that the agents perform actions, work and make decisions, the coordination among them is one of the critical issues for effective task implementation. The task implementation consists of two main aspects, i.e. task allocation and workload balancing. It was discussed that the position of agent in the network can significantly influence the system performance [15]. Centrality measures are one of the most important tools for measuring the importance of a vertex position in the complex networks. Among the many vertices and edges that form a network, some play a crucial role and are considered to be central within the network structure. Note, that the task allocation can be based on such centralities, where the probability that a task is allocated to the agent monotonically increases with its centrality value.

In this work we consider the *shortest path betweenness centrality* [10] which relies on the idea that information flows along shortest paths through a network. It is defined to quantify the importance to which a vertex or an edge has control over pair-wise connections between other vertices, based on the assumption that the importance of connections is equally divided among all shortest paths for each pair. Brandes [5] gives a comprehensive survey and compares most recent variants of betweenness centrality. Some of them are the *proximal target betweenness* introduced by Borgatti, the *bounded-distance betweenness* defined by Borgatti and Everett [4] and the *edge betweenness*, as a natural extension of betweenness to edges, firstly discussed by Anthonisse [1]. Other variants of the general betweenness centrality are fundamentally different in their

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calculation and they are mostly based on the idea that a realistic betweenness measure should include non-shortest paths in addition to the shortest ones. The two most known measures of this type are the *flow-betweenness centrality* [3,6,11] and the *random walk betweenness centrality* [6,19]. Moreover, there are two important generalizations of the betweenness centrality, namely *group betweenness* [9,21,22] which identifies groups of vertices which have collective influence in a network, as well as *routing betweenness* [8] which considers network flows created by arbitrary loop-free routing strategies.

A feature shared by many real-world networks is *power-law degree distribution*, i.e. vertices with small degrees are most frequent and the fraction of highly connected vertices decreases, but is not zero. Similarly, many scale-free networks are also characterized by *power-law betweenness distribution*, which means that a few vertices will be overloaded if the load of a vertex is defined as the betweenness centrality value. Thus, the skewed betweenness distribution in real-world networks leads to imbalanced work allocation across vertices. In this study, our aim is to measure the extent to which the workload can be equalized between the vertices within the network based on local information. For this reason, we introduce degree-weighted measure based on betweenness centrality.

2. Preliminaries

In this section we briefly introduce the important terms underlying our work, give the definition of a new invariant, as well as the results related to our study.

Let G be an undirected simple graph comprising a set $V(G)$ of vertices together with a set $E(G)$ of edges. Usually, we denote by n the number of vertices of G, i.e. $n = |V(G)|$. Let $\delta(G)$ denote the minimum degree of a graph G. Note that $\delta(G) \geq 1$, since we restrict our observation only to connected graphs. A *path* between two vertices $u, v \in V(G)$, denoted (u, v)-path, is a sequence of vertices starting at u and ending at v, i.e. $us_1s_2 \cdots s_kv$, such that from each of its vertices there is an edge to the next vertex in the sequence. The *length* of an (u, v)-path is the number of edges it contains. The *distance* between u and v, $d(u, v)$, is defined as the minimum length of any (u, v)-path. Such a path always exists, since we consider only connected graphs.

Denote by $\sigma(u, v)$ the number of shortest (u, v)-paths, and let $\sigma(u, v|ij)$ be the number of shortest (u, v)-paths passing through some edge ij. In the original definition given by Freeman [10], the betweenness centrality of a vertex v is defined by the probabilistic number of shortest paths between pairs of vertices that uses vertex v. In the present paper, we use a slightly different way of defining betweenness centrality measure and use the idea proposed in [7]. We first define betweenness for edges and later extend the definition to vertices by summation. To each edge $ij \in E(G)$ we associate the probabilistic number of shortest paths b_{ij} in which it is involved:

$$b_{ij} = \sum_{\substack{u,v \in V(G)\\ u<v}} \frac{\sigma(u, v|ij)}{\sigma(u, v)} \tag{1}$$

and extend this definition to the definition of the betweenness centrality for vertices as follows:

$$B_i = \sum_{j \in N(i)} b_{ij} \tag{2}$$

for every vertex $i \in V(G)$, where $N(i)$ denotes the set of neighbours of a vertex i in a graph G. Caporossi et al. [7] proved the upper and lower bounds of extremal B_i values:

$$n - 1 \leq \max_{i \in V(G)} B_i \leq (n-1)^2,$$

$$n - 1 \leq \min_{i \in V(G)} B_i \leq \begin{cases} \dfrac{n^2-1}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n^2}{4} & \text{if } n \text{ is even.} \end{cases}$$

As noted in [7], this centrality measure is closely related to Freeman's betweenness centrality, denoted by B_i^F. The relation between these two measures is given by $B_i = 2B_i^F + n - 1$, where $n = |V(G)|$. Note that Freeman's centrality for vertex i accounts only for paths, where i is a middle vertex, while B_i accounts also for paths, where i is an end vertex. Another important relation emphasized in [7] is relation between the Wiener index [23] and the betweenness centrality. The *Wiener index* of a graph G, denoted by $W(G)$, is the sum of distances between all (unordered) pairs of vertices of G, i.e.

$$W(G) = \sum_{u,v \in V(G)} d(u, v),$$

and it is related to the betweenness centrality as $2W(G) = \sum_{i \in V(G)} B_i$. In the class of 2-connected and 2-edge-connected graphs of order n, the cycle C_n has maximal Wiener index [20],

$$W(C_n) = \begin{cases} \dfrac{n(n^2-1)}{8} & \text{if } n \text{ is odd,} \\ \dfrac{n^3}{8} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$