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Trichotomy for integer linear systems based on their sign patterns☆

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ABSTRACT

In this paper, we consider solving the integer linear systems, i.e., given a matrix $A \in \mathbb{R}^{m\times n}$, a vector $b \in \mathbb{R}^m$, and a positive integer d, to compute an integer vector $x \in D^n$ such that $Ax \geq b$ or to determine the infeasibility of the system, where m and n denote positive integers, $\mathbb{R}$ denotes the set of reals, and $D = \{0, 1, \dots, d-1\}$. The problem is one of the most fundamental NP-hard problems in computer science.

For the problem, we propose a complexity index η which depends only on the sign pattern of A. For a real γ, let ILS(γ) denote the family of the problem instances I with $\eta(I) = \gamma$. We then show the following trichotomy:

- ILS(γ) is solvable in linear time, if $\gamma < 1$,
- ILS(γ) is weakly NP-hard and pseudo-polynomially solvable, if $\gamma = 1$,
- ILS(γ) is strongly NP-hard, if $\gamma > 1$.

This, for example, includes the previous results that Horn systems and two-variable-per-inequality (TVPI) systems can be solved in pseudo-polynomial time.

1. Introduction

Integer linear systems

In this paper, we consider solving the integer linear systems, i.e., given a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, a vector $b \in \mathbb{R}^m$, and a positive integer d, to compute an integer vector $x \in D^n$ such that $Ax \geq b$ or to determine the infeasibility of the system, where m and n denote positive integers, $\mathbb{R}$ denotes the set of reals, and $D = \{0, 1, \dots, d-1\}$. We denote the problem by ILS. The ILS problem is one of the most fundamental and important problems in computer science, and has been studied extensively from both theoretical and practical points of view [19,30].

The ILS problem is strongly NP-hard, but several (semi-)tractable subclasses are known to exist. For example, the problem can be solved in polynomial time, if n is bounded by some constant [25], or A is totally unimodular and b is integral [16]. Moreover, the ILS problem for the following three subclasses can be solved in pseudo-polynomial time: (1) m is bounded by some constant and A is integral [28], (2) A has at most two nonzero elements per row [2,15] (such a system is called

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TVPI system), and (3) A is Horn [12,33] (i.e., each row of A contains at most one positive element). It is also known that the problem is weakly NP-hard, even if m is bounded by some constant or A has at most two nonzero elements per row and Horn (also called monotone) [23]. The best known bounds for TVPI and Horn systems respectively require $O(md)$ time [2,15] and $O(n^2md)$ time [12], where the algorithms in [2,15] are both based on [10].

For unit linear systems, i.e., when $A \in \{0, -1, +1\}^{m\times n}$, it is known that the problem is still strongly NP-hard, but for TVPI systems, it can be solved in $O(nm)$ time [24] and $O(n \log n + m)$ time [31], and for Horn systems, it can be solved in $O(n^2m)$ time [9,32].[1] Finally, for the difference constraint systems, i.e., the monotone unit systems, it is known that the problem is equivalent to the negative cycle detection in network theory and can be solved in $O(nm)$ time [3,11,27] and $O(\sqrt{n}m \log C)$ time [13], where C denotes the maximum absolute value of the negative elements in b.

A complexity index for integer linear systems

In this paper, we introduce a complexity index η for the ILS problem, which sharply distinguishes among the classes of *easy*, *semi-hard* and *hard* integer linear systems. The idea of this index originates from the work by Boros et al. [5], which introduced a complexity index for the Boolean satisfiability problem (SAT), which distinguishes between the classes of easy and hard SAT instances. Our index η is a generalization of theirs to integer linear systems, since SAT problems can be represented as integer linear systems with unit matrices $A \in \{0, -1, +1\}^{m\times n}$, which will be discussed in Section 2.

For a real a, its sign is defined as

$$\mathrm{sgn}(a) = \begin{cases} + & (a > 0) \\ 0 & (a = 0) \\ - & (a < 0), \end{cases} \tag{1}$$

and the sign of a real matrix $A \in \mathbb{R}^{m\times n}$ is the matrix $\mathrm{sgn}(A) \in \{0, -, +\}^{m\times n}$ which is obtained from A by replacing each element by its sign.

For example, for a matrix

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 4 & 2 & -5 \end{pmatrix}, \tag{2}$$

we have

$$\mathrm{sgn}(A) = \begin{pmatrix} + & - & 0 \\ + & + & - \end{pmatrix}. \tag{3}$$

Given an instance $I = (A, b, d)$ of the ILS problem, index $\eta(I)$ is the optimal value of the following linear programming problem.

$$\begin{array}{lll} \text{minimize} & Z & \\ \text{subject to} & \displaystyle\sum_{j:\mathrm{sgn}(a_{ij})=+} \alpha_j + \sum_{j:\mathrm{sgn}(a_{ij})=-} (1-\alpha_j) \le Z & (i = 1, \ldots, m) \\ & 0 \le \alpha_j \le 1 & (j = 1, \ldots, n). \end{array} \tag{4}$$

We note that neither numerical information of A nor any information about b or d is used for our index $\eta(I)$, and it depends *only* on $\mathrm{sgn}(A)$, i.e., two problem instances I and I' have $\eta(I) = \eta(I')$ if the corresponding matrices have the same sign patterns. Thus we sometimes call $\eta(I)$ *index* of A.

Main results obtained in this paper

For a nonnegative real γ, let $\mathrm{ILS}(\gamma)$ denote the family of the problem instances I with $\eta(I) = \gamma$. For unit linear systems, i.e., when $A \in \{0, -1, +1\}^{m\times n}$, $\mathrm{UILS}(\gamma)$ is defined analogously. We then have the following main result.

Theorem 1. *For a nonnegative real γ with $\mathrm{ILS}(\gamma) \neq \emptyset$ (equivalently, $\mathrm{UILS}(\gamma) \neq \emptyset$), we have the following three cases:*

(1) *$\mathrm{ILS}(\gamma)$ is solvable in linear time, if $\gamma < 1$,*
(2) *$\mathrm{ILS}(\gamma)$ is weakly NP-hard and pseudo-polynomially solvable, if $\gamma = 1$,*
(3) *$\mathrm{ILS}(\gamma)$ is strongly NP-hard, if $\gamma > 1$.*

Moreover, $\mathrm{UILS}(\gamma)$ is polynomially solvable, if $\gamma = 1$, and strongly NP-hard, if $\gamma > 1$.

[1] Note that Horn unit system without upper bounds on the variables, i.e., $d = +\infty$, is called Horn system and Horn Constraint System in [9] and [32], respectively.