Discrete Applied Mathematics 200 (2016) 95–107

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Complete characterization of bicyclic graphs with minimal Kirchhoff index[☆]

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ARTICLE INFO

Article history:
Received 8 September 2014
Received in revised form 29 April 2015
Accepted 2 July 2015
Available online 29 July 2015

Keywords:
Resistance distance
Kirchhoff index
Bicyclic graphs
Extremal graphs
Θ-type graphs
Wiener index

ABSTRACT

The resistance distance between any two vertices of a graph G is defined as the network effective resistance between them if each edge of G is replaced by a unit resistor. The Kirchhoff index $K(G)$ is the sum of the resistance distances between all the pairs of vertices in G. A bicyclic graph is a connected graph whose number of edges is exactly one more than its number of vertices. In this paper, we completely characterize the bicyclic graphs of order $n \geq 4$ with minimal Kirchhoff index and determine bounds on the Kirchhoff index of bicyclic graphs. This improves and extends some earlier results.

1. Introduction

All graphs considered in this paper are simple and undirected. As usual, denote by P_n, C_n and S_n a path, cycle and star with n vertices, respectively. A bicyclic graph $G = (V, E)$ is a connected simple graph which satisfies $|E(G)| = |V(G)| + 1$. Exactly, there are two basic types of bicyclic graphs: ∞-type graphs and Θ-type graphs. Obviously, a bicyclic graph contains either two or three cycles.

For convenience, we employ some notations. Let $\mathcal{B}(n)$ be the set of connected n-vertex bicyclic graphs. A Θ-graph is the union of three internally disjoint paths with two common end vertices, see Fig. 1. A Θ-type graph, denoted by $\Theta_n^{p,q,m}$, is a union of three internally disjoint paths $P_p : v_0v_1 \ldots v_p$, $P_q : u_0(=v_p)u_1 \ldots u_q(=v_0)$, $P_{m+1} : v_0w_1 \ldots w_mv_p$, of length $p, q, m+1$ respectively with common end vertices, and the trees $T_{v_i}(0 \leq i \leq p-1)$, $T_{u_j}(0 \leq j \leq q-1)$, $T_{w_k}(1 \leq k \leq m)$ are rooted at v_i, u_j, w_k, respectively, see Fig. 2. We say a tree T trivial if $|V(T)| = 1$, i.e., T is a single vertex. Let a Θ-type graph $S_n^{p,q,m}$ denote the graph obtained from a Θ-type graph $\Theta_n^{p,q,m}$ by attaching all of its pendent vertices to one of the two common end vertices. For other undefined notations and terminology from graph theory, the readers are referred to [5].

Let G be a graph with vertices labeled $1, 2, \ldots, n$. Recall that the conventional distance between vertices v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path between them. The famous Wiener index $W(G)$ [27] is the sum of distances

☆ Partially supported by National Natural Science Foundation of China (No. 11471016, 11401004 and 11371028), Anhui Provincial Natural Science Foundation (No. 1408085QA03), Natural Science Foundation of Anhui Province of China (No. KJ2013B105).

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http://dx.doi.org/10.1016/j.dam.2015.07.001

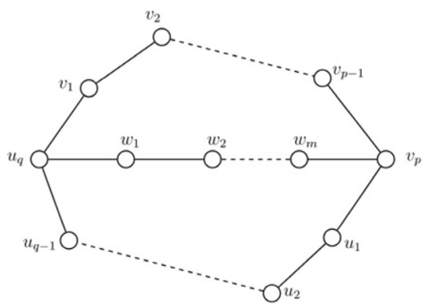

Fig. 1. An illustration of the Θ-graph.

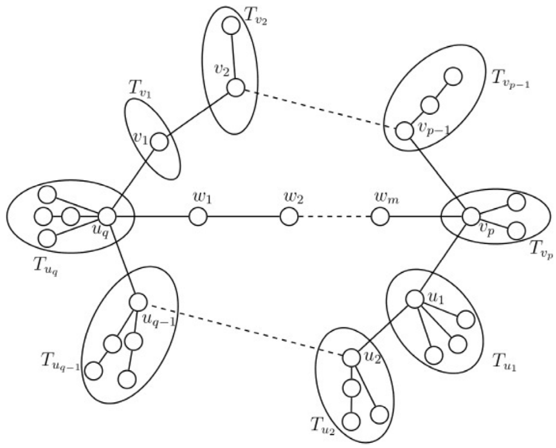

Fig. 2. An illustration of the Θ-type graph.

between all pairs of vertices, i.e.,

$$W(G) = \frac{1}{2} \sum_{(v_i, v_j) \in V \times V} d_G(v_i, v_j).$$

Klein and Randić [14] introduced a new distance function named resistance distance based on electrical network theory, the resistance distance between vertices i and j, denoted by $r_G(v_i, v_j)$, is defined to be the effective electrical resistance between them if each edge of G is replaced by a unit resistor [14]. Analogous to Wiener index, the famous distance-based topological index as Kirchhoff index $K(G)$ [14], is defined as the sum of resistance distances between all pairs of vertices in G, i.e.,

$$K(G) = \frac{1}{2} \sum_{(v_i, v_j) \in V \times V} r_G(v_i, v_j).$$

Actually, the shortest path might be imagined to be more relevant when there is corpuscular communication along edges between two vertices in graphs G, whereas the resistance distance might be imagined to be more relevant when the communications are wavelike or fluid-like. Then the fact that the chemical communication in the molecules is rather wavelike suggests the utility of this concept in chemistry [10]. Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry, graph theory, etc. [1,7,11,28,29,38–40]. In 1996, Gutman and Mohar [12] obtained the famous result by which a relation is established between the Kirchhoff index and the Laplacian spectrum: $K(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix of a connected graph G with n vertices. R.B. Bapat et al. has provided a simple method for computing the resistance distance in [2]. J.L. Palacios [19–24] studied the resistance distance and the Kirchhoff index on connected undirected graphs with probability methods. E. Bendito et al. [3] formulated the Kirchhoff index based on discrete potential theory. M. Bianchi et al. obtained the upper and lower bounds for the Kirchhoff index $K(G)$ of an arbitrary simple connected graph G by using a majorization technique [4]. The Kirchhoff index is also a structure descriptor. However, it is difficult to calculate the resistance distances and Kirchhoff index of graphs. Hence it makes sense to compute the special classes of graphs. We know that the Kirchhoff index and the Wiener index of trees coincide. It is possible to study the Kirchhoff index of topological structures containing cycles, for instance, Deng et al. [37] obtained the second maximal and minimal Kirchhoff index of unicyclic graphs. Wang et al. [25] obtained the first three minimal Kirchhoff indices among cacti. Yang et al. [33] studied Kirchhoff index of unicyclic graphs with given girth and