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Bounds on the differentiating-total domination number of a tree

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ABSTRACT

Given a graph $G = (V, E)$ with no isolated vertex, a subset S of V is called a total dominating set of G if every vertex in V is adjacent to a vertex in S. A total dominating set S is called a differentiating-total dominating set if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a differentiating-total dominating set of G is the differentiating-total domination number of G, denoted by $\gamma_t^D(G)$. We show that, for a tree T of order $n \geq 3$ and diameter d having l leaves and s support vertices, $\frac{3(d+1)}{5} \leq \gamma_t^D(T) \leq n - \frac{2(d-2)}{5}$ and $\frac{6}{11}(n + 1 + \frac{l}{2} - s) \leq \gamma_t^D(T) \leq \frac{3(n+l)}{5}$. Moreover, we characterize the extremal trees achieving these bounds.

1. Introduction

The concept of a locating-total dominating set and a differentiating-total dominating set in a graph was introduced in [3,6]. The problem of placing monitoring devices in a system such that every site (including the monitors themselves) in the system is adjacent to a monitor can be modeled by total domination in graphs. Applications where it is also important that if there is a problem in a device, its location can be uniquely identified by the set of monitors, can be modeled by a combination of total domination and locating in graphs. In this paper, we consider differentiating-total domination in trees.

This paper will follow the notation and terminology defined in [4,5]. Let $G = (V, E)$ be a graph of order n with no isolated vertex. For a vertex v in G, the set $N(v) = \{u \in V \mid uv \in E\}$ is called the *open neighborhood* of v and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. For a subset $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ is the *open neighborhood* of S and $N[S] = N(S) \cup S$ is the *closed neighborhood* of S. A subset S of V is called a *dominating set* (DS) of G if $N[S] = V$ and S is a *total dominating set* (TDS) of G if $N(S) = V$. A TDS S is a *locating-total dominating set* (LTDS) if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and S is a *differentiating-total dominating set* (DTDS) if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a LTDS (or DTDS) of G is the *locating-total domination number* (or *differentiating-total domination number*) of G and denoted by $\gamma_t^L(G)$ (or $\gamma_t^D(G)$). A LTDS (or DTDS) of cardinality $\gamma_t^L(G)$ (or $\gamma_t^D(G)$) is called a $\gamma_t^L(G)$*-set* (or $\gamma_t^D(G)$*-set*).

Given a graph $G = (V, E)$, the *degree* of v in G, denoted by $d(v)$ or $d_G(v)$, is equal to $|N(v)|$. A vertex of degree one is a *leaf* and the edge incident with a leaf is known as a *pendent edge*. A vertex adjacent to a leaf is a *support vertex* and a support vertex adjacent to at least two leaves is a *strong support vertex*. We will use $L(G)$ and $S(G)$ to denote the set of leaves and support vertices of G, respectively. For arbitrary two vertices u and v in G, the *distance* between u and v, denoted by $d(u, v)$, is the number of edges in a shortest path joining u and v. If there is no such path, then we define $d(u, v) = \infty$. The *diameter*

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of G is the maximum distance among all pairs of vertices of G, denoted by $diam(G)$. If A and B are two disjoint subsets of V, then $[A, B] = \{uv \in E(G) | u \in A, v \in B\}$. For a subset S of V, we use $G[S]$ to denote the subgraph induced by S. Let G and H be two disjoint graphs. The *disjoint union* of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we write kG_1 for $G_1 + \cdots + G_k$.

A path of order n is P_n. A star of order n is denoted by S_n. A tree is called a *double star* $S(p, q)$, if it is obtained from S_{p+2} and S_{q+1} by identifying a leaf of S_{p+2} with the center of S_{q+1}, where $p, q \geq 1$. Given a graph $G = (V, E)$, the *corona* of G, *cor(G)*, is a graph obtained from G by adding a pendent edge to each vertex of G.

Differentiating-total domination in trees has been studied in [1,6]. In this paper, we continue the study of it. We show that, for a tree T of order $n \geq 3$ and diameter d having l leaves and s support vertices, $\frac{3(d+1)}{5} \leq \gamma_t^D(T) \leq n - \frac{2(d-2)}{5}$ and $\frac{6}{11}(n + 1 + \frac{l}{2} - s) \leq \gamma_t^D(T) \leq \frac{3(n+l)}{5}$. Moreover, we characterize the extremal trees achieving these bounds.

2. Lower bounds on the differentiating-total domination number of a tree

The differentiating-total domination number of P_n was given in [6].

Theorem 1 *([6]). For $n \geq 3$,*

$$\gamma_t^D(P_n) = \begin{cases} \left\lceil \dfrac{3n}{5} \right\rceil & \text{if } n \not\equiv 3 \pmod 5, \\ \left\lceil \dfrac{3n}{5} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 5. \end{cases}$$

The following result gives a lower bound of $\gamma_t^D(T)$ involving diameter. Suppose n and d are two positive integers with $d+1 \equiv 0 \pmod 5$ and $d+1 \leq n \leq \frac{6(d+1)}{5}$. Let ξ_1 be the family of trees of order n and diameter d that can be obtained from a path $P = x_1\text{–}x_2\text{–} \ldots \text{–}x_{d+1}$ of length d by adding $n - d - 1$ isolated vertices such that each new vertex is adjacent only to vertices in $\bigcup_{i=1}^{(d+1)/5}\{x_{5i-2}\}$ and the resulting tree has no strong support vertices.

Theorem 2. *Suppose T is a tree of order $n \geq 3$ and diameter d, then $\gamma_t^D(T) \geq \frac{3(d+1)}{5}$ and the equality holds if and only if $T \in \xi_1$.*

Proof. We proceed by induction on the order n. If $n = 3$, then $T = P_3$ and $\gamma_t^D(T) = 3 > \frac{3(d+1)}{5}$ by Theorem 1.

Assume that every tree T' of order $3 \leq n' < n$ and diameter d' satisfies $\gamma_t^D(T') \geq \frac{3(d'+1)}{5}$. Let T be a tree of order $n > 3$ and diameter d. If T is a path, then $\gamma_t^D(T) \geq \frac{3n}{5} = \frac{3(d+1)}{5}$ by Theorem 1.

Now suppose T is not a path. Let $P = x_1\text{–}x_2\text{–} \ldots \text{–}x_{d+1}$ be a path of length d in T and $v \in V(P)$ such that $d(v) \geq 3$. Let u be a vertex in $V(T) - V(P)$ such that $d(v, u)$ is maximum. Then $u \in L(T)$. Let $N(u) = \{w\}$ and $T' = T - u$. Then $n' = n - 1 \geq 3$ and $d' = d$. By the inductive hypothesis, $\gamma_t^D(T') \geq \frac{3(d'+1)}{5} = \frac{3(d+1)}{5}$.

Let D be a $\gamma_t^D(T)$-set of T. Since $w \in S(T)$, $w \in D$. Let T_w be the component in $T[D]$ containing w. If $u \notin V(T_w)$, then D is a DTDS of T'. Now assume $u \in V(T_w)$. If $|V(T_w)| \geq 4$, then $D \setminus \{u\}$ is a DTDS of T'. So we assume $|V(T_w)| = 3$.

If $d(w) = 2$, then $V(T_w) = N[w] = \{u, w, z\}$ for some vertex z. Let $t \in N(z) \setminus \{w\} \neq \emptyset$, then $t \notin D$ and $(D \setminus \{u\}) \cup \{t\}$ is a DTDS of T'. If $d(w) \geq 3$, then there is a vertex $t \in N(w) - D$. Thus, $(D \setminus \{u\}) \cup \{t\}$ is a DTDS of T'. In each case, we have $|D| \geq \gamma_t^D(T')$. This completes the proof of $\gamma_t^D(T) \geq \frac{3(d+1)}{5}$.

If $T \in \xi_1$, it is easy to verify that $D = \bigcup_{i=1}^{(d+1)/5}\{x_{5i-3}, x_{5i-2}, x_{5i-1}\}$ is a DTDS of T. Thus, $\gamma_t^D(T) \leq \frac{3(d+1)}{5}$. Since $\gamma_t^D(T) \geq \frac{3(d+1)}{5}$, we have $\gamma_t^D(T) = \frac{3(d+1)}{5}$.

Conversely, suppose T is a tree of order $n \geq 3$ and diameter d satisfying $\gamma_t^D(T) = \frac{3(d+1)}{5}$. Then $d + 1 \equiv 0 \pmod 5$. Let $P = x_1\text{–}x_2\text{–} \ldots \text{–}x_{d+1}$ be a path of length d in T and D a $\gamma_t^D(T)$-set of T. For $i = 1, 2, \ldots, \frac{d+1}{5}$, let T_i be the component of $T - \bigcup_{j=1}^{(d-4)/5}\{x_{5j}x_{5j+1}\}$ containing the vertex x_{5i} and $P_i = x_{5i-4}\text{–}x_{5i-3}\text{–}x_{5i-2}\text{–}x_{5i-1}\text{–}x_{5i}$ be a subpath of P (we define $\bigcup_{j=1}^{(d-4)/5}\{x_{5j}x_{5j+1}\} := \emptyset$ if $d = 4$). Since D is a DTDS of T, $|D \cap V(T_i)| \geq 3$. Thus, $|D| \geq 3(d+1)/5$. As $|D| = \gamma_t^D(T) = 3(d+1)/5$, we obtain $|D \cap V(T_i)| = 3$ for $i = 1, 2, \ldots, \frac{d+1}{5}$. We will show that $D \cap V(T_i) = \{x_{5i-3}, x_{5i-2}, x_{5i-1}\}$ for $i = 1, 2, \ldots, \frac{d+1}{5}$.

Fact 1. $|D \cap V(P_i)| = 3$ for $i = 1, 2, \ldots, \frac{d+1}{5}$.

Proof of Fact 1. Suppose there is an $i_0 \in \{1, 2, \ldots, \frac{d+1}{5}\}$ such that $|D \cap V(P_{i_0})| = 1$. Assume $D \cap V(P_{i_0}) = \{x\}$. Since every component of $T[D]$ has at least three vertices and $|V(T_{i_0}) \cap D| = 3$, there are two vertices y and z in $(V(T_{i_0}) - V(P_{i_0})) \cap D$ with either $y \in N(x)$ and $z \in N(y)$, or $\{y, z\} \subseteq N(x)$. If $x \in \{x_{5i_0-4}, x_{5i_0-3}\}$ (resp. $x \in \{x_{5i_0-1}, x_{5i_0}\}$), then $N(x_{5i_0-1}) \cap D = \emptyset$ (resp. $N(x_{5i_0-3}) \cap D = \emptyset$), a contradiction. If $x = x_{5i_0-2}$, then $N[x_{5i_0-3}] \cap D = N[x_{5i_0-1}] \cap D = \{x_{5i_0-2}\}$, a contradiction. Thus, $|D \cap V(P_i)| \geq 2$ for $i = 1, 2, \ldots, \frac{d+1}{5}$.