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An extension of the Motzkin–Straus theorem to non-uniform hypergraphs and its applications

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ABSTRACT

In 1965, Motzkin and Straus established a remarkable connection between the order of a maximum clique and the Lagrangian of a graph and provided a new proof of Turán's theorem using the connection. The connection of Lagrangians and Turán densities can be also used to prove the fundamental theorem of Erdős–Stone–Simonovits on Turán densities of graphs. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems and suggested by Johnston and Lu. In this paper, we attempt to explore the applications of Lagrangian method in determining Turán densities of non-uniform hypergraphs. We first give a definition of the Lagrangian of a non-uniform hypergraph, then give an extension of the Motzkin–Straus theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Applying it, we give an extension of the Erdős–Stone–Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Our approach follows from the approach in Keevash's paper Keevash (2011).

1. Introduction and main results

Turán problems on uniform hypergraphs have been actively studied. In 1965, Motzkin and Straus provided a new proof of Turán's theorem based on a remarkable connection between the order of a maximum clique and the Lagrangian of a graph in [9]. In fact, the connection of Lagrangians and Turán densities can be used to give another proof of the fundamental theorem of Erdős–Stone–Simonovits on Turán densities of graphs in [8]. This type of connection aroused interests in the study of Lagrangians and Motzkin–Straus type results of uniform hypergraphs. For example, in [13], Talbot studied properties of Lagrangians of uniform hypergraphs; in [10], Rota Bulò and Pelillo gave an extension of the Motzkin–Straus theorem to uniform hypergraphs. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems (see [4] and [5]). In this paper, we attempt to explore the applications of Lagrangian method in determining Turán densities of non-uniform hypergraphs. We first give a definition of the Lagrangian of a non-uniform hypergraph, then give an extension of the Motzkin–Straus theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Applying it, we give an extension of the Erdős–Stone–Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices.

A hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where every edge in $E(H)$ is a subset of $V(H)$. The set $R(H) = \{|F| : F \in E\}$ is called the set of *edge types* of H. We also say that H is a $R(H)$-graph. For example, if $R(H) = \{1, 2\}$, then we say that H is a $\{1, 2\}$-graph. If all edges have the same cardinality k, then H is a k-uniform hypergraph.

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A 2-uniform hypergraph is called a graph. A hypergraph is non-uniform if it has at least two edge types. For any $k \in R(H)$, the *level hypergraph* H^k is the hypergraph consisting of all edges with k vertices of H. We write H_n^R for a hypergraph H on n vertices with $R(H) = R$. An edge $\{i_1, i_2, \ldots, i_k\}$ in a hypergraph is simply written as $i_1 i_2 \cdots i_k$ throughout the paper.

For an integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a set V and integer i, let $\binom{V}{i}$ be the family of all i-subsets of V. The complete hypergraph K_n^R is a hypergraph on n vertices with edge set $\bigcup_{i \in R} \binom{[n]}{i}$. For example, $K_n^{\{k\}}$ is the complete k-uniform hypergraph on n vertices. $K_n^{[k]}$ is the non-uniform hypergraph with all possible edges of cardinality at most k. The complete graph on n vertices $K_n^{\{2\}}$ is also called a clique. We also let $[k]^T$ represent the complete T-graph on vertex set $[k]$.

Let us briefly review the Turán problem on uniform hypergraphs. For a given r-uniform hypergraph F and positive integer n, let $\text{ex}(n, F)$ be the maximum number of edges an r-uniform hypergraph on n vertices can have without containing F as a subgraph. By a standard averaging argument of Katona, Nemetz, and Simonovits in [7], $\frac{\text{ex}(n,F)}{\binom{n}{r}}$ decreases as n increases, therefore $\lim_{n\to\infty} \frac{\text{ex}(n,F)}{\binom{n}{r}}$ exists. This limit is called the *Turán density* of F and denoted by $\pi(F)$. Turán's theorem [14] implies that $\pi(K_l^{\{2\}}) = 1 - \frac{1}{l-1}$. The fundamental result in extremal graph theory due to Erdős–Stone–Simonovits generalizes Turán's theorem and it says that for a graph F with chromatic number $\chi(F)$ where $\chi(F) \geq 3$, then $\pi(F) = 1 - \frac{1}{\chi(F)-1}$. However, we know quite little about Turán density of r-uniform hypergraphs for $r \geq 3$ though some progress has been made.

A useful tool in extremal problems of uniform hypergraphs (graphs) is the Lagrangian of a uniform hypergraph (graph).

Definition 1.1. Let G be an r-uniform graph with vertex set $[n]$ and edge set $E(G)$. Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. For $\vec{x} = (x_1, x_2, \ldots, x_n) \in S$, define

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

The Lagrangian of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

Motzkin and Straus in [9] shows that the Lagrangian of a graph is determined by the order of its maximum cliques.

Theorem 1.1 (*Motzkin and Straus [9]*)**.** *If G is a graph in which a largest clique has order l, then $\lambda(G) = \lambda(K_l^{\{2\}}) = \lambda([l]^{(2)}) = \frac{1}{2}(1 - \frac{1}{l})$.*

This connection provided another proof of Turán's theorem. More generally, the connection of Lagrangians and Turán densities can be used to give another proof of the Erdős–Stone–Simonovits result(see Keevash's survey paper [8]). In 1980s, Sidorenko [11] and Frankl and Füredi [2] developed the method of applying Lagrangians in determining hypergraph Turán densities. More applications of Lagrangians can be found in [3,12] and [8]. Very recently, the study of Turán densities of non-uniform hypergraphs have been motivated by the study of extremal poset problems [4,5]. A generalization of the concept of Turán density to a non-uniform hypergraph was given in [6] by Johnston and Lu.

For a non-uniform hypergraph G on n vertices, the Lubell function of G is defined to be

$$h_n(G) = \sum_{k \in R(G)} \frac{|E(G^k)|}{\binom{n}{k}}.$$

Given a family of hypergraph $\mathcal{F}$ with common set of edge-types R, the Turán density of $\mathcal{F}$ is defined to be

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \max\{h_n(G) : |V(G)| = n, R(G) \subseteq R, \text{ and } G \text{ contains no subgraph in } \mathcal{F}\}.$$

In [6], it is shown that $\max\{h_n(G) : |V(G)| = n, R(G) \subseteq R, \text{ and } G \text{ contains no subgraph in } \mathcal{F}\}$ decreases as n increases, hence the limit exists. For a hypergraph F, $\pi(\{F\})$ is simply written as $\pi(F)$.

Definition 1.2. For a hypergraph H with n vertices and positive integers $s_1, s_2, \ldots, s_n$, the blowup of H is a new hypergraph (V, E), denoted by $H(s_1, s_2, \ldots, s_n)$, satisfying

1. $V = \bigcup_{i=1}^n V_i$ is a union of n pairwise disjoint sets, where $|V_i| = s_i$;
2. E is obtained by replacing each edge $F \in E(H)$ by a complete $|F|$-partite $|F|$-uniform hypergraph with partition sets $\cup_{i \in F} V_i$.

Remark 1.2. For a non-uniform hypergraph G on n vertices, the blowup of G has the following property:

$$h_{nt}(G(t, t, \ldots, t)) \leq h_n(G).$$

This can be verified easily by a direct calculation.