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# Global Roman domination in graphs

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ABSTRACT

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. A set $S \subseteq V$ is a *global dominating set* if $S$ dominates both $G$ and its complement $\overline{G}$. The global domination number $\gamma_g(G)$ of a graph $G$ is the minimum cardinality of $S$. We define a *global Roman dominating function* on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ such that $f$ is an RDF for both $G$ and its complement $\overline{G}$. The *weight* of a global Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a global Roman dominating function on a graph $G$ is called the *global Roman domination number* of $G$ and denoted by $\gamma_{gR}(G)$. In this paper, we initiate a study of this parameter.



## 1. Introduction

In the 4th century A.D., when the Roman Empire was under attack during the period of Emperor Constantine the Great, he had the requirement that an army or a legion could be sent from its home to defend a neighbouring location only if there was a second army which would stay and protect the home. Thus there are two types of armies, stationary and travelling. Each vertex with no army must have a neighbouring vertex with a travelling army. Stationary armies then dominate their own vertices and a vertex with two armies are dominated by its stationary army and its open neighbourhood is dominated by the travelling army. The objective, of course, is to minimize the total number of legions needed. The problem generalizes to arbitrary graphs.

Cockayne et al. [2] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. For a real valued function $f : V \to R$, the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$ and for $S \subseteq V, f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in $G$; that is $\gamma_R(G) = min\{w(f)/f$ is an RDF in $G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. Roman domination in graphs has been studied in [2–5,8–16,19]. This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled “Defend the Roman Empire!” [18].

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to Chartand and Lesniak [1]. All graphs in this paper are assumed to be connected with at least two vertices. A set $S$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in $G$ and a dominating set $S$ of minimum cardinality is called a $\gamma$-set of $G$. The literature on domination and its variations in graphs has been surveyed and detailed in the two books by Haynes et al. [6,7].

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E. Sampathkumar [17] defined a global dominating set as follows: A set $S \subseteq V$ is a *global dominating set* if $S$ dominates both $G$ and its complement $\overline{G}$ and a global dominating set of minimum cardinality is called a $\gamma_g(G)$-set. The *global domination number* $\gamma_g(G)$ is the minimum cardinality of a global dominating set in $G$.

In this paper, we extend the idea of global domination to a Roman domination function as follows: We define a *global Roman dominating function* (GRDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ such that $f$ is an RDF for both $G$ and its complement $\overline{G}$. The *weight* of a global Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a global Roman dominating function on a graph $G$ is called the *global Roman domination number* of $G$ and denoted by $\gamma_{gR}(G)$.

## 2. Notations

The *degree* of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G(v)$ or simply $\deg(v)$ when the graph is clear from the context. A vertex of degree zero in $G$ is called an *isolated vertex*, while a vertex of degree one is called a *leaf* vertex or a *pendant vertex* of $G$. The *minimum degree* of $G$ is the minimum degree among the vertices of $G$ and is denoted by $\delta(G)$. The *maximum degree* of $G$ is defined as the maximum degree among the vertices of $G$ and is denoted by $\triangle(G)$.

In a connected graph $G$, the *distance between two vertices u* and $v$ is the number of edges in a shortest path joining $u$ and $v$ if any; and is denoted by $d(u, v)$. If $u \in V$ and $S \subset V$, then $d(u, S)$ denotes the minimum distance between $u$ and any vertex of $S$. The *eccentricity* of a vertex $v$ is $ecc(v) = max\{d(v, w); w \in V\}$. The *radius* of a graph $G$ is $\text{rad}(G) = min\{ecc(v) : v \in V\}$ and the *diameter* of the graph $G$ is $\text{diam}(G) = max\{ecc(v) : v \in V\}$.

The union of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the union of their vertex and edge sets: $G \cup H = (V_G \cup V_H, E_G \cup E_H)$. When $V_G$ and $V_H$ are disjoint, their union is referred to as the disjoint union.

For $n \geq 4$, the *wheel* $W_n$ is defined to be the graph $K_1 + C_{n-1}$ where $C_{n-1}$ is a cycle on $(n - 1)$ vertices. The central vertex in $W_n$ of degree $n - 1$ is called the *hub vertex*.

A graph $G$ is *bipartite* if the vertex set can be partitioned into two disjoint subsets $A$ and $B$ such that the vertices in $A$ are only adjacent to vertices in $B$ and vice versa. $K_{r,s}$ denotes the *complete bipartite graph* where $V = A \cup B$, $|A| = r$, $|B| = s$, $A$ and $B$ are independent sets and every vertex in $A$ is adjacent to every vertex in $B$. A complete bipartite graph of the form $K_{1,n}$ is called a *star* graph. we call the vertex of degree $n$, the head vertex.

A maximal complete subgraph of a graph $G$ is called a *clique*. The clique number $\omega(G)$ of a graph $G$ is the maximum order among the complete subgraphs of $G$.

A connected graph having no cycle is called a *tree*. A *split graph* is a graph $G = (V, E)$ whose vertices can be partitioned into two sets $X$ and $Y$ where the vertices in $X$ are independent and vertices in $Y$ form a complete graph.

A set $S$ of vertices is called *independent* if no two vertices in $S$ are adjacent.

For any set $S \subseteq V$, the *induced subgraph S* is the maximal subgraph of $G$ with vertex set $S$ and is denoted by $G[S]$.

A *support* is a vertex which is adjacent to at least one leaf vertex. A *weak support* is a vertex which is adjacent to exactly one leaf vertex. A *strong support* is a vertex which is adjacent to at least two leaf vertices.

For any vertex $v \in V$, the *open neighbourhood* of $v$ is the set $N(v) = \{u \in V / uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighbourhood* is $N[S] = N(S) \cup S$. Let $v \in S \subseteq V$. A Vertex $u$ is called a *private neighbour* of $v$ with respect to $S$ (denoted by $u$ is an $S$-pn of $v$) if $u \in N[v] - N[S - \{v\}]$. An *S-pn* of $v$ is *external* if it is a vertex of $V - S$. The set $pn(v, S) = N[v] - N[S - \{v\}]$ of all $S$-pns of $v$ is called the *private neighbourhood set* of $v$ with respect to $S$.

## 3. Properties of global Roman dominating set

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ and let $(V_0, V_1, V_2)$ be the ordered partition of $V$ induced by $f$, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. Note that there exists a one to one correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions $(V_0, V_1, V_2)$ of $V$. Thus, we will write $f = (V_0, V_1, V_2)$.

**Proposition 3.1.** *For any graph G,* $\gamma_g(G) \leq \gamma_{gR}(G) \leq 2\gamma_g(G)$.

**Proof.** Let $f = (V_0, V_1, V_2)$ be a global Roman dominating function. Since $V_2$ dominates $V_0$, $V_1 \cup V_2$ is a global Roman dominating set of $G$. Now

$$\begin{aligned}\gamma_g(G) &\leq |V_1 \cup V_2| \\ &\leq |V_1| + |V_2| \\ &< |V_1| + 2|V_2| \\ &= \gamma_{gR}(G)\end{aligned}$$

Now let $S$ be a $\gamma_g(G)$-set. Define $V_0 = V - S$; $V_1 = \emptyset$; $V_2 = S$ and let $f = (V_0, V_1, V_2)$. Since $V_2$ dominates $V_0$, $f$ is a global Roman dominating function and $\gamma_{gR}(G) \leq f(V) = 2|S| = 2\gamma_g(G)$. Hence $\gamma_{gR}(G) \leq 2\gamma_g(G)$ and therefore $\gamma_g(G) \leq \gamma_{gR}(G) \leq 2\gamma_g(G)$. □