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Fractional Thue chromatic number of graphs

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ABSTRACT

This paper introduces the concept of the fractional Thue chromatic number of graphs and studies the relation between the fractional Thue chromatic number and the Thue chromatic number. We determine the fractional Thue chromatic number of all paths, all trees with no vertices of degree two, and all cycles, except C_{10}, C_{14}, C_{17}. As a consequence, we prove that if G is a path or a tree with no degree two vertices, then its fractional Thue chromatic number equals its Thue chromatic number. On the other hand, we show that there are trees and cycles whose fractional Thue chromatic numbers are strictly less than their Thue chromatic numbers.

1. Introduction

A sequence $x_1x_2\cdots x_{2\ell}$ of even length is a *repetition* if the first half is identical to the second half, i.e., $x_i = x_{i+\ell}$ for all $1 \leqslant i \leqslant \ell$. A sequence is *nonrepetitive* if it does not contain a string of consecutive entries that forms a repetition. In 1906, Thue [12] constructed an infinite nonrepetitive sequence using only three symbols. Thue's result is of fundamental importance in many branches of mathematics and has been re-discovered many times in different context. It leads to the birth of Symbolic Dynamics, and has many generalizations and important applications in distinct fields of mathematics and computer science [1].

Alon, Grytczuk, Hałuszczak, Riordan [2] introduced the notion of nonrepetitiveness to graph colouring: a colouring c of the vertices of a graph G is *nonrepetitive* if for any path $v_1, \ldots, v_{2\ell}$ in G with an even number of vertices, the sequence $c(v_1), \ldots, c(v_{2\ell})$ is not a repetition. The *Thue chromatic number* of G is the least integer $\pi(G)$ such that there exists a nonrepetitive colouring c of G using $\pi(G)$ colours. With this notation, Thue's result says $\pi(P_\infty) = 3$ (it can be verified easily that $\pi(P_n) \geq 3$ for $n \geq 4$).

The Thue chromatic number of graphs has been studied in many papers [2,4,6,11,8]. Upper and lower bounds for the Thue chromatic number of many classes of graphs are obtained in the literature. It was proved in [4] that all trees have Thue chromatic number at most 4, and this upper bound is sharp. This result is generalized in [11] where it is shown that any partial k-tree has Thue chromatic number at most 4^k. The Thue chromatic number of cycles are determined by Currie [6].

Theorem 1 ([6]). *Let C_n be a cycle on $n \geq 3$ vertices. If $n = 5, 7, 9, 10, 14, 17$, then $\pi(C_n) = 4$. Otherwise $\pi(C_n) = 3$.*

For graphs of bounded maximum degree, it is proved in [2] that graphs G of maximum degree d have $\pi(G) \leq Cd^2$, for some constant C. The constant C was shown to be at most $2e^{16}$ in [2], and was reduced to 16 by Grytczuk [8], and to 1 (ignoring lower order terms) by Dujmović, Joret, Kozik and Wood [7]. On the other hand, there are graphs G of maximum degree d with $\pi(G) \geq C'\frac{d^2}{\log d}$ for some positive constant C'.

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The Thue chromatic number of the lexicographic product of graphs is studied in [10]. It is shown in [10] that for every tree T, $\pi(T[K_k]) \leq 4k$ and this bound is sharp, where $T[K_k]$ is the lexicographic product of T and K_k which is obtained from T by replacing each vertex with a copy of K_k. For sufficiently long paths P, $\pi(P[K_k]) \geq 3k + \lfloor k/2 \rfloor$, and for $k > 2$, $\pi(P[E_k]) = 2k + 1$, where E_k is the graph with k vertices and no edges.

Harant and Jendrol' [9] defined the *facial Thue chromatic number* of a plane graph G as the minimum number of colours needed to colour the vertices of G so that the colours assigned to the vertices of any facial path form a nonrepetitive sequence. They conjectured that the facial Thue chromatic numbers of plane graphs are bounded from above by a constant. This conjecture was proved by Barát and Czap [3].

This paper introduces the fractional Thue chromatic number of graphs.

Definition 1. A *k-tuple n-colouring* of a graph G is a mapping $f : V(G) \to \binom{[n]}{k}$ which assigns to each vertex v a set $f(v)$ of k colours from the set $[n] = \{1, 2, \ldots, n\}$ of n colours. A k-tuple n-colouring f of G is *nonrepetitive* if for any path $v_1, v_2, \ldots, v_{2\ell}$ in G with an even number of vertices, for any $c_i \in f(v_i)$, $1 \leq i \leq 2\ell$, the string $c_1, c_2, \ldots, c_{2\ell}$ is not a repetition. The *fractional Thue chromatic number* $\pi_f(G)$ of G is defined as

$$\pi_f(G) = \inf\{n/k : \text{there exists a } k\text{-tuple nonrepetitive } n\text{-colouring of G}\}.$$

It follows from the definition that $\pi_f(G) \leq \pi(G)$ for any graph G. We are interested in the following two natural questions:

1. For which graphs $\pi_f(G) = \pi(G)$?
2. How big can be the difference $\pi(G) - \pi_f(G)$?

Although the Thue chromatic number of graphs has been studied in many papers, there are not many graphs G for which $\pi(G)$ are known. The determination of the fractional Thue chromatic number of graphs seems to be even more difficult. In this paper, we study the fractional Thue chromatic number of paths, cycles and trees. We determine the fractional Thue chromatic number of paths, trees with no vertices of degree two, cycles of length not equal to 10, 14, 17. As consequences of our results, we have $\pi_f(G) = \pi(G)$ if G is a path, or a tree with no vertices of degree two. For cycles, we show that $\pi_f(C_n) < \pi(C_n)$ if $n \in \{7, 9, 10, 14, 17\}$ and $\pi_f(C_n) = \pi(C_n)$ otherwise. We also show that there are trees T with $\pi_f(T) = 3.5 < \pi(T)$. In the last section, we shall propose a few questions.

2. Paths and trees

To determine the fractional Thue chromatic number of paths is easy. Let P_n be the path on n vertices. It is easy to see that $\pi_f(P_n) = 2$ for $2 \leq n \leq 3$. Now we show that for $n \geq 4$, $\pi_f(P_n) = 3$.

Lemma 1. *Let $P = v_1v_2v_3v_4$ be a path of 4 vertices. If c is a k-tuple nonrepetitive colouring of P, then either $c(v_1) \cap c(v_3) = \emptyset$ or $c(v_2) \cap c(v_4) = \emptyset$.*

Proof. Otherwise, let $a \in c(v_1) \cap c(v_3)$ and $b \in c(v_2) \cap c(v_4)$. Then, there is a repetition $abab$ on the path $v_1v_2v_3v_4$, a contradiction. ∎

Corollary 1. *If $n \geq 4$, then $\pi_f(P_n) = 3$.*

Proof. By Lemma 1, any k-tuple nonrepetitive colouring of P_n has 3 consecutive vertices whose colour sets are pairwise disjoint. So the total number of colours is at least $3k$, i.e., $\pi_f(P_n) \geq 3$. On the other hand, $\pi_f(P_n) \leq \pi(P_n) \leq 3$, so $\pi_f(P_n) = 3$ for $n \geq 4$. ∎

For trees, there is no easy algorithm that determines their Thue chromatic number. However, for trees without degree two vertices, their Thue chromatic number is determined by the following theorem. The eccentricity of a vertex v in G is $\rho(v) = \max\{d_G(v, u) : u \in V(G)\}$, and *radius* of G is $\mathrm{rad}(G) = \min\{\rho(v) : v \in V(G)\}$.

Theorem 2 ([4]). *Let T be a tree in which no vertex is of degree two. Then $\pi(T) \leq 3$ if and only if $rad(T) \leq 4$.*

We shall extend this result to fractional Thue chromatic number and prove the following result.

Theorem 3. *Assume T is a tree with no vertices of degree two.*

$$\pi_f(T) = \begin{cases} 1, & \text{if } T = K_1 \\ 2, & \text{if } \mathrm{rad}(T) = 1, \text{ i.e., } T \text{ is a star} \\ 3, & \text{if } 2 \leq \mathrm{rad}(T) \leq 4 \\ 4, & \text{if } \mathrm{rad}(T) \geq 5. \end{cases}$$