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Combinatorial algorithms with performance guarantees for finding several Hamiltonian circuits in a complete directed weighted graph[☆]

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ABSTRACT

In this paper we present two new polynomial algorithms for the asymmetric version of the m-Peripatetic Salesman Problem (m-APSP) which consists in finding m edge-disjoint Hamiltonian circuits of extremal total weight in a complete weighted digraph. The first algorithm solves the asymmetric 2-PSP on maximum. Its approximation ratio is equal to 2/3. The second algorithm deals with the minimization version of the asymmetric m-PSP on random instances. For this algorithm conditions for asymptotically exactness are presented.

1. Introduction

The m-Peripatetic Salesman Problem (m-PSP) is a natural generalization of the classical Traveling Salesman Problem (TSP). In m-PSP we need to find m edge-disjoint Hamiltonian cycles of maximum or minimum total weight in a complete weighted graph (the symmetric version of the problem) or digraph (the asymmetric version). The problem has been introduced in [24] by Krarup. It was investigated both for deterministic or random instances, for arbitrary [1,19], metric [2,5] and Euclidean weight functions [4,14] which can be different or common for all circuits, and for special cases of the problem where edge weights belong to a given interval or a finite set of numbers [3,16–18,20]. In particular, for the symmetric version of the problem the following results were established. In [1,19] two polynomial algorithms for the maximization variant of 2-PSP with approximation ratios 3/4 and 7/9 were designed. The authors of [3,7,16] present a series of polynomial approximation algorithms for the minimization variant of 2-PSP with edge weights 1 and 2, where the weight function is common for both Hamiltonian cycles, while in [18,20] the same problem was studied for two different weight functions. In this case two polynomial algorithms with approximation ratios 7/5 and 4/3 were developed. For the Euclidean maximum-weight m-PSP an asymptotically exact algorithm with time complexity $O(n^3)$ was designed [4].

Applications include the design of patrol tours [6] where it is often important to assign a set of edge-disjoint tours to the watchman in order to avoid constant repetition of the same tour and thus enhance security. De Kort [11] cites a network

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design application where, in order to protect the network from link failure, several edges-disjoint cycles must be determined. He also mentions a scheduling application of the 2-PSP where each job must be processed twice by the same machine but technological constraints prevent the repetition of identical job sequences.

De Brey and Volgenant [8] identify several easy cases of 2-PSP. De Kort developed lower and upper bounds for 2-PSP, through branch-and-bound algorithms, and showed that the problem of finding two edge-disjoint Hamiltonian circuits is NP-complete [9–11]. This result implies that 2-PSP with identical weight functions is NP-hard both in the maximization and minimization variants. The problem is also NP-hard for the case of different weight functions [5].

So, the efforts of most researchers are concentrated on finding cases for which the problem can be solved in polynomial time and developing polynomial approximation algorithms for TSP and m-PSP. A review of the most significant obtained results in this area can be found in [15,21]. Mostly, these results focus on the symmetric variants of the problem.

Routing problems in the case of asymmetric transport networks or graphs are of current interest as well. Theoretical results in this area turn out to be rather difficult to obtain. For example, one of the best approximation algorithms known for the asymmetric maximum TSP is Kaplan et al.'s [23] algorithm with guaranteed ratio 2/3, while for the symmetric maximum TSP an algorithm with the approximation ratio 3/4 has been known since 1984 [21], and several more recent papers present algorithms with even better approximation ratios 25/33 [22] and 7/9 [25]. Note that the algorithm for the asymmetric TSP from [23] has non-combinatorial nature and runs by solving a large sophisticated linear program.

An *Asymmetric m-Peripatetic Salesman Problem* (m-APSP) is a problem of finding m edge-disjoint Hamiltonian circuits $H_1, \ldots, H_m$ in G with maximum or minimum total edge weight $w^* = w(H_1) + \cdots + w(H_m)$. Clearly, m-APSP is a generalization of the asymmetric TSP (ATSP). For the m-APSP, $m \geq 2$, no approximation algorithms with performance guarantees were known. As was mentioned above, for the maximization variant of ATSP, Kaplan et al. [23] developed an approximation algorithm with guaranteed ratio 2/3 through linear programming tools. In the first part of this paper we present a similar result for a generalization of that problem, namely 2-APSP on maximum (2-APSP-max), and by using purely combinatorial tools. More precisely, we construct an approximation algorithm $A_{2/3}$ for 2-APSP-max with guaranteed ratio 2/3 and cubic running-time. In the second part of the paper we present an $O(mn^2)$ algorithm $\widetilde{A}$ for the m-APSP with a different weight function for each salesman and study the performance of this algorithm on complete graphs with random weights uniformly distributed in some positive real numbers interval. We establish some conditions under which algorithm $\widetilde{A}$ is asymptotically exact.

2. Preliminary definitions

We consider a complete n-vertex digraph $G = G(V, E)$ with the vertex set $V = V(G)$ and the edge set $E = E(G)$; $w : E \to \mathbf{R}_+$ is an arbitrary non-negative weight function of the edges of G.

For a vertex $v \in V$ in a digraph H we use the following notation:

- $d^+_H(v) = d^+(v)$ — *indegree* (the number of incoming edges of v in H).
- $d^-_H(v) = d^-(v)$ — *outdegree* (the number of outgoing edges of v in H).
- $d_H(v) = d(v) = d^+(v) + d^-(v)$ — *degree* of v.
- An *oriented 2-factor* in H is a collection of edge-disjoint circuits covering all vertices of H.
- A *partial tour* in H is a collection of vertex-disjoint directed paths covering all vertices of H (which may include so called *singletons*, i.e. paths with just one vertex). For a partial tour T, by $|T|$ we denote the number of edges of T, while by $P(T)$ and $p(T)$ — the set and the number of all paths in T respectively. Clearly, $|T| + p(T) = |V(H)|$ for any partial tour T in H.
- A *bipartite model* of a digraph H is a bipartite undirected graph D with parts $V = V(H)$ and V', where V' is the set of all duplicates of vertices of H, and $\{X, Y'\} \in E(D) \Leftrightarrow (X, Y) \in E(H)$.

3. Main result for 2-APSP-max: algorithm $A_{2/3}$

In this section we present algorithm $A_{2/3}$ for 2-APSP-max with arbitrary non-negative weight function.

Let $w(OPT)$ be the weight of the optimal solution of 2-APSP-max. Our goal is the following:

Theorem 1. *Algorithm $A_{2/3}$ described below finds two edge-disjoint Hamiltonian circuits H_1, H_2 in a complete weighted digraph G with the property $w(H_1) + w(H_2) \geq \frac{2}{3} w(OPT)$. The running-time of the algorithm is $O(n^3)$.*

3.1. A sketch of algorithm $A_{2/3}$

If $n \leq 15$, then we find (with brute force) an optimal solution of the problem, i.e. a pair of edge-disjoint Hamiltonian circuits H_1, H_2 in G with maximum total weight.

Suppose $n \geq 16$. The idea behind the algorithm is the following:

Phase 1. First we find the maximum weight sub-graph G_4 of G such that the in-degree and the out-degree of each vertex of G_4 are equal to 2. This can be done in cubic running-time by Gabow's algorithm [12]. Clearly, the weight of G_4 is at least as large as the weight of the optimal solution of 2-APSP-max.