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About the minimum mean cycle-canceling algorithm

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ABSTRACT

This paper focuses on the resolution of the capacitated minimum cost flow problem on a network comprising n nodes and m arcs. We present a method that counts imperviousness to degeneracy among its strengths, namely the *minimum mean cycle-canceling* algorithm (MMCC). At each iteration, primal feasibility is maintained and the objective function strictly improves. The goal is to write a uniform and hopefully more accessible paper which centralizes the ideas presented in the seminal work of Goldberg and Tarjan (1989) as well as the additional paper of Radzik and Goldberg (1994) where the complexity analysis is refined. Important properties are proven using linear programming rather than constructive arguments.

We also retrieve Cancel-and-Tighten from the former paper, where each so-called phase which can be seen as a group of iterations requires $O(m \log n)$ time. MMCC turns out to be a strongly polynomial algorithm which runs in $O(mn)$ phases, hence in $O(m^2 n \log n)$ time. This new complexity result is obtained with a combined analysis of the results in both papers along with original contributions which allows us to enlist Cancel-and-Tighten as an acceleration strategy.

1. Introduction

This paper addresses the resolution of the capacitated minimum cost flow problem (CMCF) on a network defined by n nodes and m arcs. We present the *minimum mean cycle-canceling* algorithm (MMCC). The seminal work of Goldberg and Tarjan [13], as presented in the book of Ahuja et al. [1], as well as the paper of Radzik and Goldberg [17], where the complexity analysis is refined, are the underlying foundations of this document. The current literature states that MMCC is a strongly polynomial algorithm that performs $O(m^2 n)$ iterations, a tight bound, and runs in $O(m^3 n^2)$ time.

While Goldberg and Tarjan [13] present Cancel-and-Tighten as a self-standing algorithm, we feel it belongs to the realm of acceleration strategies incidentally granting the reduction of the theoretical complexity. Our understanding is that this strategy can be shared at any level of the complexity analysis. Indeed, its very construction aims to assimilate the so-called notion of *phase* which can be seen as a group of iterations. This strategy exploits an approximation scheme to manage this assimilation and as such nevertheless necessitates a careful analysis. We propose a new approximation structure which allows us to reduce the global runtime to $O(m^2 n \log n)$. It is namely the product of a refined analysis that accounts for $O(mn)$ phases, each one requiring $O(m \log n)$ time.

The reader should view this work as much more than a synthesis. It is the accumulation of years of research surrounding degeneracy that led us to realize the ties with theories drafted some forty years ago. We not only hope to clarify the behavior

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of the minimum mean cycle-canceling algorithm but also provide strong insights about the ins and outs of its idiosyncrasies and more importantly establish a solid unified framework against which we can rest current and future work. On that note, let us underline the linear programming mindset which simplifies the construction of one of the most important parts of the algorithm, namely the pricing problem. The justification of some of its properties also benefit from straightforward implications provided by that mindset. Some fundamental properties of network problems are also incorporated throughout the text which sometimes facilitate if not, certainly enlighten, the comprehension of the proofs presented by the listed authors.

The paper is organized as follows. The elaboration of MMCC takes place in Section 2 where the combination of the so-called *residual network* along with optimality conditions give birth to a pricing problem which is put to use in an iterative process. Section 3 analyzes its complexity which is decomposed in two parts: the *outer loop* and the *bottleneck*. Although the latter comes at the very last, it acts as the binding substance of the whole paper. It is indeed where the behavior of the algorithm can be seen at a glance alongside the justification for the significance of the aforementioned *phases*. This is followed by the conclusion in Section 4.

2. Minimum mean cycle-canceling algorithm

Consider the formulation of CMCF on a directed graph $G = (N, A)$, where N is the set of n nodes associated with an assumed balanced set $b_i,\ i \in N$, of supply or demand defined respectively by a positive or negative value such that $\sum_{i\in N} b_i = 0$, A is the set of m arcs of cost $\mathbf{c} := [c_{ij}]_{(i,j)\in A}$, and $\mathbf{x} := [x_{ij}]_{(i,j)\in A}$ is the vector of bounded flow variables:

$$\begin{aligned} z^* := \quad & \min \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i, \qquad [\pi_i] \qquad && \forall i \in N \\ & 0 \le \ \ell_{ij} \le x_{ij} \le u_{ij}, && \forall (i,j) \in A, \end{aligned} \tag{1}$$

where $\boldsymbol{\pi} := [\pi_i]_{i\in N}$ is the vector of dual variables, also known as node potentials. When right-hand side $\mathbf{b} := [b_i]_{i\in N}$ is the null vector, formulation (1) is called a *circulation* problem.

Let us enter the world of network solutions with a fundamental proposition whose omitted proof traditionally relies on a constructive argument. It is so rooted in the network design that, case in point, straightforward derivatives are used throughout this document.

Proposition 1 (*Ahuja et al. [1, Theorem 3.5 and Property 3.6]*)**.** *Any feasible solution* $\mathbf{x}$ *to* (1) *can be represented as a combination of paths and cycles flows (though not necessarily uniquely) with the following properties:*

(a) *Every directed path with positive flow connects a supply node to a demand node; at most* $n + m$ *directed paths and cycles have non-zero flow among which at most* m *cycles.*
(b) *In the case of a circulation problem, by definition there are no supply nor demand nodes, which means the representation can be restricted to at most* m *directed cycles.*

This section derives MMCC, devised to solve instances of CMCF, in the following manner. Section 2.1 defines the corner stone of the resolution process, namely the residual network. Whether its inception goes back to the optimality conditions or its usage came as an afterthought is an enigma for which we have no answer. Either way, the latter are introduced thereafter and pave the way for the pricing problem in Section 2.2. Section 2.3 exhibits the algorithmic process which is ultimately information sharing between a control loop and a pricing problem. The former ensures primal feasibility while the latter provides a strictly improving direction at each iteration. Section 2.4 illustrates the behavior of the algorithm on the maximum flow problem.

2.1. Residual network and optimality conditions

The residual network takes form with respect to a feasible flow $\mathbf{x}^0 := [x^0_{ij}]_{(i,j)\in A}$ and is denoted $G(\mathbf{x}^0) = (N, A(\mathbf{x}^0))$. As eloquently resumed in Fig. 1, each arc $(i,j) \in A$ is replaced by two arcs representing upwards and downwards possible flow variations:

- arc (i,j) with cost $d_{ij} = c_{ij}$ and residual flow $0 \le y_{ij} \le r^0_{ij} := u_{ij} - x^0_{ij}$;
- arc (j,i) with cost $d_{ji} = -c_{ij}$ and residual flow $0 \le y_{ji} \le r^0_{ji} := x^0_{ij} - \ell_{ij}$.

Denote $A' := \{(i,j) \cup (j,i) \mid (i,j) \in A\}$ as the complete possible arc support of any residual network. The residual network $G(\mathbf{x}^0)$ consists of only the residual arcs, i.e., those with strictly positive residual capacities, that is, $A(\mathbf{x}^0) := \{(i,j) \in A' \mid r^0_{ij} > 0\}$. The combination of the current solution $\mathbf{x}^0$ along with the optimal marginal flow computed on the residual network is optimal for the original formulation. Indeed, the residual network with respect to $\mathbf{x}^0$ corresponds to the change of variables $x_{ij} = x^0_{ij} + (y_{ij} - y_{ji}),\ \forall (i,j) \in A$. Observe that traveling in both directions would be counterproductive and can be simplified