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First-passage-time location function: Application to determine first-passage-time densities in diffusion processes

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Abstract

A time-dependent function, namely the First-Passage-Time Location function, is introduced in the context of the study of first-passage-times. From this function, a strategy is developed in order to solve numerically the Volterra integral equation of the second kind verified by the first-passage-time densities for diffusion processes. The proposed procedure provides the advantages in the application of quadrature methods in terms of an appropriate choice of the integration step, as well as an outstanding reduction in the computational cost. Some examples are developed showing the validity of that strategy as well as the computational advantages.

1. Introduction

The first-passage-time (f.p.t.) problem for stochastic processes is the study of the time instant at which the process first reaches a critical level or threshold or, in general, it traverses a time-dependent boundary. This problem arises in many contexts. For example, in neurobiology, an action potential, or nerve impulse, is produced when the voltage at a particular place on a neuron reaches a threshold. In a growth context in livestock installations, the study of the size of individuals over time is fundamental in selecting specimens for consumption. In economics it is important to know the time at which a certain variable, like the consumer prices index or the gross national product, achieves a certain value.

Many works have focused on this subject. Among others, the work of Ricciardi et al. (1999) contains a selection of papers in a biological context, whereas in Gutiérrez et al. (1999) an application is made to the economic field.

The study of procedures for obtaining the density function of the f.p.t. through time-dependent boundaries for diffusion processes has covered several phases. As regards the homogeneous case, Ricciardi et al. (1983) proved that the density verifies a Volterra integral equation of the second kind. Buonocore et al. (1987) studied the kernel of this equation for the Wiener and Ornstein–Uhlenbeck processes, and they found conditions for its regularization. Finally, Giorno et al. (1988, 1989) generalized the regularization process of the kernel to the class of homogeneous diffusion processes. Concerning the non-homogeneous case, a first generalization of the integral equation was made by

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considering some special kinds of diffusion processes (see Gutiérrez et al. (1991, 1994, 1995, 1997)). Later, Gutiérrez et al. (1997) extended their results and showed their validity for the class of non-homogeneous diffusion processes. More recently, Di Nardo et al. (2001) proved that this result is also valid, in general, for Gauss–Markov processes.

Nevertheless, and apart from some particular processes and boundaries, closed-form solutions for the integral equation are not available. For this reason, in the cases without explicit solutions, numerical procedures are needed. The most usual methods are based on numerical quadrature procedures, as proposed by Buonocore et al. (1987) on the basis of the composite trapezoid method. General procedures for Volterra integral equations of the second kind are currently under study, for example in research carried out by Maleknejad and Shahrezaee (2004), Maleknejad et al. (2007), and Saberi-Nadjafi and Heidari (2007a,b).

However, when considering the problem of approximate f.p.t. densities, the location of the time variable can avoid some disadvantages that may appear in the application of these kinds of procedures. For instance, invalid approximations (caused by a poor choice of the integration step) or an unnecessarily high computational cost (due to different causes as an inappropriate initial time, an unnecessarily small integration step or an unsuitable stopping rule).

In this sense, we introduce the First-Passage-Time Location (*FPTL*) function. It will allows us to determine some time instants that provide useful information in order to obtain, by means of the mentioned numerical schema, good approximate densities with a smaller computational cost. Obviously, the strategy employed here focuses on the numerical solution of the Volterra integral equation. In this paper we deal with diffusion processes but the procedure can also be applied for other process whose f.p.t. densities verify this equation (like Gauss–Markov processes).

Section 2 illustrates some of the disadvantages previously cited, justifying the need for a strategy in the application of numerical procedures to solve the integral equation. The *FPTL* function is defined in Section 3, that also exposes the procedures to determine from it the information needed to apply the numerical algorithms in a more efficient way. Finally, in Section 4 some applications of the procedure are proposed. Concretely, we consider the problem of obtaining the f.p.t. density through a constant boundary for the lognormal diffusion process, showing that the procedure leads to good approximations. In addition, the computational advantages are shown by means of the Gompertz diffusion process (introduced by Gutiérrez et al. (2007)) through the study of the f.p.t. through a constant boundary.

2. Approach to the problem. Justification of the *FPTL* function

Let $\{X(t); t_0 \leq t \leq T\}$ be a diffusion process defined on a real interval I and with infinitesimal moments $A_1(x,t)$ and $A_2(x,t)$. For this process, we consider the f.p.t. through a time-dependent boundary $S(t)$, conditioned to $X(t_0) = x_0$, that is the time variable

$$T_{S(t),x_0} = \begin{cases} \underset{t \geq t_0}{\text{Inf}}\{t : X(t) > S(t) | X(t_0) = x_0\} & \text{if } x_0 < S(t_0) \\ \underset{t \geq t_0}{\text{Inf}}\{t : X(t) < S(t) | X(t_0) = x_0\} & \text{if } x_0 > S(t_0), \end{cases} \tag{1}$$

whose density function is denoted by $g(S(t), t|x_0, t_0)$.

When no conditioning to a fixed initial value is considered, the f.p.t. depends on the random variable $X(t_0)$. In such a case, definition (1) can be generalized in the following form

$$T_{S(t),X(t_0)} = \begin{cases} \underset{t \geq t_0}{\text{Inf}}\{t : X(t) > S(t)\} & \text{if } X(t_0) < S(t_0) \\ \underset{t \geq t_0}{\text{Inf}}\{t : X(t) < S(t)\} & \text{if } X(t_0) > S(t_0). \end{cases} \tag{2}$$

In this last case, and if J denotes the range of variation of $X(t_0)$, the density of (2) can be obtained from the family of densities $\{g(S(t), t|x_0, t_0), x_0 \in J - \{S(t_0)\}\}$ by means of the expression

$$g(S(t), t) = \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{S(t_0)-\epsilon} g(S(t), t|x_0, t_0) f_{X(t_0)}(x_0)\, \mathrm{d}x_0 + \int_{S(t_0)+\epsilon}^{+\infty} g(S(t), t|x_0, t_0) f_{X(t_0)}(x_0)\, \mathrm{d}x_0 \right], \tag{3}$$

where $f_{X(t_0)}$ is the density function of $X(t_0)$. Therefore, this case can be reduced to the former and for this we focus on the formulation given by (1).