Available online at www.sciencedirect.com
ScienceDirect
Computational Statistics & Data Analysis 52 (2007) 1143 – 1148
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

On the estimation of the linear relation when the error variances are known

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Available online 28 June 2007

Abstract

The problem of consistent estimation in measurement error models in a linear relation with not necessarily normally distributed measurement errors is considered. Three possible estimators which are constructed as different combinations of the estimators arising from direct and inverse regression are considered. The efficiency properties of these three estimators are derived and the effect of non-normally distributed measurement errors is analyzed. A Monte-Carlo experiment is conducted to study the performance of these estimators in finite samples.

Keywords: Measurement error models; Ultrastructural model; Non-normal errors

1. Introduction

In a linear measurement error model, the parameters can be estimated consistently only when some additional information besides the data set is available. There are various ways in which such additional information can be employed; (see, e.g., Cheng and Van Ness, 1999; Fuller, 1987). Among them, application of the knowledge of all or one of the measurement error variances is the most prominent approach.

We consider three different combinations of the direct and inverse adjusted least squares (LS) estimators. They are modelled after analogous combinations found in the literature, where, however, they have been constructed from non-adjusted direct and inverse LS estimators. Sokal and Rohalf (1981) considered the geometric mean of these two estimators (which they call the technique of reduced major axis) and Aaronsom et al. (1986) work with the arithmetic mean. In addition, the slope parameter may be estimated by the slope of the line that bisects the angle between the direct and inverse regression lines; see, e.g., Pierce and Tully (1988). While all these estimators are not consistent (although they possibly reduce the bias inherent in their constituent direct and inverse LS estimators), the present paper constructs consistent estimators by using error adjusted direct and inverse LS rather than non-adjusted direct and inverse LS estimators. A simple question then arises: which out of these suggested estimators is better under what conditions. This question has been partly dealt with in Dorff and Gurland (1961), but for a model with replicated observations and unknown error variances.

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0167-9473/$ - see front matter
doi:10.1016/j.csda.2007.06.018

The efficiency properties of all the estimators under consideration are expressed as functions of the reliability ratios associated with study and explanatory variables, (see, Gleser, 1992, 1993). The asymptotic properties of the estimators are derived when the measurement errors are not necessarily normally distributed.

The plan of our presentation is as follows. In Section 2, we describe a linear model with measurement errors and present the estimators of the slope parameter when the error variances are known. Section 3 analyzes the asymptotic properties of the estimators when the underlying error distributions are not necessarily normal. The finite sample properties of the proposed estimators under different types of distributions of measurement errors are studied through a Monte-Carlo experiment in Section 4. Some concluding remarks are offered in Section 5.

2. Model specification and the estimators

Consider a linear measurement error model in which the variables are related by the linear relation

$$Y_j = \alpha + \beta X_j \quad (j = 1, 2, \ldots, n), \tag{1}$$

where Y_j and X_j denote the true but unobserved values of the study and explanatory variables. The observed values y_j and x_j are expressible as $y_j = Y_j + u_j$ and $x_j = X_j + v_j$, respectively, where u_j and v_j denote the associated measurement errors.

We assume that $X_1, X_2, \ldots, X_n$ are independent (not necessarily identically distributed) random variables such that $\text{plim}_{n\to\infty}\bar{X}$ and $\text{plim}_{n\to\infty}(1/n)\sum(X_j - \bar{X})^2 > 0$ exist, which are denoted by μ_X and σ_X^2, respectively. The measurement errors $u_1, u_2, \ldots, u_n$ are assumed to be independent and identically distributed with mean 0, variance σ_u^2, third moment $\gamma_{1u}\sigma_u^3$ and fourth moment $(\gamma_{2u}+3)\sigma_u^4$. Similarly, the errors $v_1, v_2, \ldots, v_n$ are assumed to be independent and identically distributed with mean 0, variance σ_v^2, third moment $\gamma_{1v}\sigma_v^3$ and fourth moment $(\gamma_{2v}+3)\sigma_v^4$. Further, the random variables (X_j, u_j, v_j) are assumed to be jointly independent.

It may be noted that this model comprises the ultrastructural model, see Dolby (1976), which in turn contains the structural and the functional model as special cases.

Consistent estimation of the parameters α and β in the relationship (1) with the help of given data (x_j, y_j), $j=1, \ldots, n$, is possible only when some additional information is available. This additional information, let us suppose, specifies the error variances σ_u^2 and σ_v^2. We can then estimate the slope parameter β consistently by using the knowledge of either of the two error variances. This provides the following well-known estimators of β:

$$b_d = \frac{s_{xy}}{s_{xx} - \sigma_v^2} \quad \text{and} \quad b_i = \frac{s_{yy} - \sigma_u^2}{s_{xy}},$$

where $s_{xx} = (1/n)\sum(x_j - \bar{x})^2$, $s_{yy} = (1/n)\sum(y_j - \bar{y})^2$, $s_{xy} = (1/n)\sum(x_j - \bar{x})(y_j - \bar{y})$, $\bar{x} = (1/n)\sum x_j$, and $\bar{y} = (1/n)\sum y_j$. An estimator using the knowledge of both the error variances is given by

$$b_p = t_p + \left(t_p^2 + \frac{\sigma_u^2}{\sigma_v^2}\right)^{1/2} \quad t_p = \frac{1}{2s_{xy}}\left(s_{yy} - \frac{\sigma_u^2}{\sigma_v^2}s_{xx}\right). \tag{2}$$

We can combine the two basic estimators b_d and b_i in various ways. One possibility is to estimate the slope parameter β by the geometric mean of the estimators b_d and b_i:

$$b_g = \text{sign}(s_{xy})|b_d b_i|^{1/2}. \tag{3}$$

Similarly, we may estimate β by the arithmetic mean of b_d and b_i:

$$b_m = \tfrac{1}{2}(b_d + b_i). \tag{4}$$

Another interesting estimator of β is

$$b_b = t_b + (t_b^2 + 1)^{1/2} \quad t_b = \frac{b_d b_i - 1}{b_d + b_i}, \tag{5}$$

which is the slope of the line that bisects the angle between the two regression lines specified by b_d and b_i.