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The degree resistance distance of cacti

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ABSTRACT

Graph invariants, based on the distances between the vertices of a graph, are widely used in theoretical chemistry. The *degree resistance distance* of a graph G is defined as $D_R(G) = \sum_{\{u,v\}\subseteq V(G)}[d(u) + d(v)]R(u, v)$, where $d(u)$ is the degree of the vertex u, and $R(u, v)$ the resistance distance between the vertices u and v. Let $Cact(n; t)$ be the set of all cacti possessing n vertices and t cycles. The elements of $Cact(n; t)$ with minimum degree resistance distance are characterized.

1. Introduction

The graphs considered in this paper are finite, loopless, and contain no multiple edges. Given a graph G, let $V(G)$ and $E(G)$ be, respectively, its vertex and edge sets. The ordinary distance $d(u, v) = d_G(u, v)$ between the vertices u and v of the graph G is the length of the shortest path between u and v.

The *Wiener index* $W(G)$ is the sum of ordinary distances between all pairs of vertices, that is, $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v)$. It is the oldest and one of the most thoroughly studied distance-based graph invariants [8,9,25].

A modified version of the Wiener index is the *degree distance* defined as $D(G) = \sum_{\{u,v\}\subseteq V(G)}[d(u) + d(v)]d(u, v)$, where $d(u) = d_G(u)$ is the degree of the vertex u of the graph G. The degree distance was also widely studied [24,1,15,6,7,18–21,31]. Tomescu [19] determined the unicyclic and bicyclic graphs with minimum degree distance. Yuan and An [31] determined the unicyclic graphs with maximum degree distance.

In 1993, Klein and Randić [16] introduced a new distance function named *resistance distance*, based on the theory of electrical networks. They viewed G as an electric network N by replacing each edge of G with a unit resistor. The resistance distance between the vertices u and v of the graph G, denoted by $R(u, v)$, is then defined to be the effective resistance between the nodes u and v in N. This new kind of distance between vertices of a graph was eventually studied in detail [16,3,4,11,12,23,30,28].

If the ordinary distance is replaced by resistance distance in the expression for the Wiener index, one arrives at the *Kirchhoff index*

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R(u, v)$$

which also has been widely studied [2,10,13,22,27,26,32].

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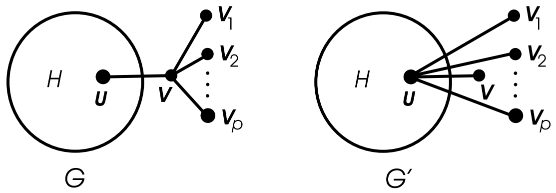

Fig. 1. The σ-transformation at v.

Similarly, if the ordinary distance is replaced by resistance distance in the expression for the degree distance, then one arrives at the *degree resistance distance* [14]:

$$D_R(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]R(u, v).$$

Palacios [17] named the same graph invariant "*additive degree–Kirchhoff index*".

In [14] some properties of D_R are determined and the unicyclic graphs with minimum and the second minimum D_R-value characterized. Bianchi et al. [5] gave some upper and lower bounds for D_R whose expressions do not depend on the resistance distances. Yang and Klein [29] gave formulae for the degree resistance distance of the subdivisions and triangulations of graphs.

A cactus is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph belongs to at most one simple cycle. Denote by $Cact(n; t)$ the set of cacti possessing n vertices and t cycles. If $G \in Cact(n; t)$, then $|E(G)| = n + t - 1$. In this paper, we determine the minimum degree resistance distance among graphs in $Cact(n; t)$ and characterize the corresponding extremal graphs.

2. Preliminaries

Let $R_G(u, v)$ denote the resistance distance between u and v in the graph G. Recall that [16] $R_G(u, v) = R_G(v, u)$ and $R_G(u, v) \geq 0$ with equality if and only if $u = v$.

For a vertex v in G, we define

$$Kf_v(G) = \sum_{u\in G} R_G(u, v) \quad \text{and} \quad D_v(G) = \sum_{u\in G} d_G(u)R_G(u, v).$$

In the above formulas and in what follows, for the sake of conciseness, instead of $u \in V(G)$ we write $u \in G$.

By the definition of $D_R(G)$, we also have

$$D_R(G) = \sum_{v\in G} d_G(v) \sum_{u\in G} R_G(u, v).$$

Lemma 1 (*[16]*). *Let G be a graph, x be a cut vertex of G and let u, v be vertices belonging to different components which arise upon deletion of x. Then $R_G(u, v) = R_G(u, x) + R_G(x, v)$.*

Lemma 2 (*[14]*). *Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V(G_1) \bigcup V(G_2) = V(G)$. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$, $m_2 = |E(G_2)|$. Then*

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2Kf_v(G_1) + 2m_1Kf_v(G_2) + (n_2 - 1)D_v(G_1) + (n_1 - 1)D_v(G_2).$$

Let v be a vertex of degree $p + 1$ in a graph G, such that $vv_1, vv_2, \ldots, vv_p$ are pendent edges incident with v, and u is the neighbor of v distinct from $v_1, v_2, \ldots, v_p$. We form a graph $G' = \sigma(G, v)$ by deleting the edges $vv_1, vv_2, \ldots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$. We say that G' is a σ-transform of G (see Fig. 1).

Lemma 3 (*[14]*). *Let $G' = \sigma(G, v)$ be a σ-transform of the graph G, $d_G(u) \geq 1$. Then $D_R(G) \geq D_R(G')$. Equality holds if and only if G is a star with v as its center.*

Let $G - v$ be the graph obtained from the graph G by deleting its vertex v and all edges incident to v.

Lemma 4. *Let u be a vertex of G such that there are p pendent vertices $u_1, u_2, \ldots, u_p$ attached to u. Let v be another vertex of G such that there are q pendent vertices $v_1, v_2, \ldots, v_q$ attached to v. Let*

$$G_1 = G - \{vv_1, vv_2, \ldots, vv_q\} + \{uv_1, uv_2, \ldots, uv_q\}$$

and

$$G_2 = G - \{uu_1, uu_2, \ldots, uu_p\} + \{vu_1, vu_2, \ldots, vu_p\}.$$

Then either $D_R(G) > D_R(G_1)$ or $D_R(G) > D_R(G_2)$.