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Improved statistical inference for the two-parameter Birnbaum–Saunders distribution

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Abstract

We develop nearly unbiased estimators for the two-parameter Birnbaum–Saunders distribution [Birnbaum, Z.W., Saunders, S.C., 1969a. A new family of life distributions. J. Appl. Probab. 6, 319–327], which is commonly used in reliability studies. We derive modified maximum likelihood estimators that are bias-free to second order. We also consider bootstrap-based bias correction. The numerical evidence we present favors three bias-adjusted estimators. Different interval estimation strategies are evaluated. Additionally, we derive a Bartlett correction that improves the finite-sample performance of the likelihood ratio test in finite samples.

Keywords: Birnbaum–Saunders distribution; Bootstrap; Bartlett correction; Bias correction; Interval estimation; Jackknife; Likelihood ratio test; Maximum likelihood estimation

1. Introduction

Birnbaum and Saunders (1969a) proposed a family of two-parameter distributions to model failure time due to fatigue under cyclic loading and the assumption that failure follows from the development and growth of a dominant crack. Desmond (1985) derived the distribution under a more general setting, using a biological model and relaxing several of the assumptions made by Birnbaum and Saunders (1969a), and Desmond (1986) explored the relationship between the Birnbaum–Saunders and inverse Gaussian distributions.

The random variable T is Birnbaum–Saunders distributed with parameters $\alpha, \beta > 0$, denoted $\mathscr{B}\text{-}\mathscr{S}(\alpha, \beta)$, if its distribution function is given by

$$F_T(t) = P(T \leqslant t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], \quad t > 0, \tag{1}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. α is a shape parameter, and as α decreases towards zero the Birnbaum–Saunders distribution approaches the normal distribution with mean β and variance τ, where $\tau \to 0$ when $\alpha \to 0$. Also, β is a scale parameter, i.e., $T/\beta \sim \mathscr{B}\text{-}\mathscr{S}(\alpha, 1)$. Additionally, β is the median of the distribution: $F_T(\beta) = \Phi(0) = 0.5$. It is noteworthy that the reciprocal property holds for the Birnbaum–Saunders distribution: $T^{-1} \sim \mathscr{B}\text{-}\mathscr{S}(\alpha, \beta^{-1})$; see Saunders (1974).

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Mann et al. (1974, p. 155) noted that, although the hazard rate implied by the Birnbaum–Saunder distribution is not increasing, the average hazard rate is nearly nondecreasing. Engelhardt et al. (1981) developed interval estimation for α considering β as an unknown nuisance parameter, and likewise for β. They also considered hypothesis testing for performing inference on the two parameters that index the distribution. Rieck and Nedelman (1991) developed a log-linear model for the $\mathcal{B}$-$\mathcal{S}(\alpha, \beta)$ distribution and showed that it can be used for accelerated life testing or to compare median lives of several populations. Achcar (1993) developed Bayesian estimation approaches for the parameters in (1) using approximations to the posterior marginal distributions of α and β. Rieck (1999) obtained the moment generating function of the sinh-normal (normal hyperbolic sine) distribution, which can be used to obtain integer and non-integer moments of $\mathcal{B}$-$\mathcal{S}(\alpha, \beta)$. Lu and Chang (1997) used bootstrap methods to construct prediction intervals for future realizations of the distribution $\mathcal{B}$-$\mathcal{S}(\alpha, \beta)$, and concluded that such intervals have good coverage when the sample contains more than 30 observations. Dupuis and Mills (1998) used robust methods to estimate the two parameters that index the Birnbaum–Saunders distribution when the sample contains outlying data. Other references related to the Birnbaum–Saunders distribution are Chang and Tang (1993, 1994), Díaz–García and Leiva–Sánchez (2005), Galea et al. (2004), Jin and Kawczak (2003), Ng et al. (2003), Owen and Padgett (1999, 2000), Rieck (1995), Wang et al. (2006) and Wu and Wong (2004). For further details on the Birnbaum–Saunders distribution, see Johnson et al. (1995).

A bias correction to the maximum likelihood estimators (MLEs) of the parameters that index the Birnbaum–Saunders distribution was proposed by Ng et al. (2003). However, their correction is ad hoc in the sense that it was obtained by "inspecting the pattern of the bias of the MLEs" in a large Monte Carlo experiment. It is not possible, thus, to guarantee that their bias-adjusted estimators are unbiased to second order.

The chief goal of our paper is to obtain modified MLEs that are nearly free of bias, in particular, modified MLEs that are unbiased to second order. That is, we remove the bias of the MLEs of α and β to order $O(n^{-1})$. This is done analytically and closed-form expressions for the second order biases of the MLEs are provided. Bootstrap-based (Efron, 1979) bias-adjusted estimators are also considered. Additionally, we numerically evaluate the finite-sample behavior of different interval estimation strategies and derive a Bartlett correction to the likelihood ratio test, thus obtaining a modified test with superior finite-sample performance.

The paper unfolds as follows. Section 2 introduces the Birnbaum–Saunders distribution, point estimation and asymptotic confidence intervals. In Section 3, we derive the second order biases of the MLEs and consider bias-correction estimators using both analytical and numerical correction schemes. We also introduce four alternative confidence intervals, three of them are bootstrap-based. In Section 4, we derive a Bartlett correction to the likelihood ratio test used to perform inference on the shape parameter of the Birnbaum–Saunders distribution. A bootstrap-based test is also considered. Numerical results from Monte Carlo simulation experiments are presented and discussed in Section 5. Two empirical examples are considered in Section 6. Finally, Section 7 concludes the paper.

2. The Birnbaum–Saunders distribution

The distribution function of the random variable T is given in (1), the corresponding density function being

$$f_T(t; \alpha, \beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right\}, \tag{2}$$

$t > 0$, $\alpha, \beta > 0$. The expected value, variance, skewness and kurtosis are, respectively,

$$\mathrm{E}(T) = \beta(1 + \tfrac{1}{2}\alpha^2), \quad \mathrm{Var}(T) = (\alpha\beta)^2(1 + \tfrac{5}{4}\alpha^2),$$

$$\mu_3 = \frac{4\alpha(11\alpha^2 + 6)}{(5\alpha^2 + 4)^{3/2}} \quad \text{and} \quad \mu_4 = 3 + \frac{6\alpha^2(93\alpha^2 + 40)}{(5\alpha^2 + 4)^2};$$

the expressions we give for the skewness and kurtosis correct those given by Johnson et al. (1995, p. 653). As noted earlier, if $T \sim \mathcal{B}$-$\mathcal{S}(\alpha, \beta)$, then $T^{-1} \sim \mathcal{B}$-$\mathcal{S}(\alpha, \beta^{-1})$. It then follows that

$$\mathrm{E}(T^{-1}) = \beta^{-1}(1 + \tfrac{1}{2}\alpha^2) \quad \text{and} \quad \mathrm{Var}(T^{-1}) = \alpha^2\beta^{-2}(1 + \tfrac{5}{4}\alpha^2).$$