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A comparative study of tests for the difference of two Poisson means

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Abstract

We investigate different test procedures for testing the difference of two Poisson means. Asymptotic tests, tests based on an approximate *p*-value method, and a likelihood ratio test are considered. Size and power performance of these tests are studied by means of Monte Carlo simulation under different settings. If one wants to control the actual significance level at or below the pre-chosen nominal level, tests based on approximate *p*-value method are the desirable candidates. If one allows tests whose actual significance levels may occasionally exceed the pre-chosen nominal level by an acceptable margin, asymptotic tests based on an unbiased estimate and constrained maximum likelihood estimate are reasonable alternatives. We illustrate these testing procedures with a breast cancer example.

Keywords: Asymptotic tests; Constrained maximum likelihood estimation; Level of significance; Likelihood ratio test; Monte Carlo simulation; Power

1. Introduction

The Poisson distribution is a mathematical rule that assigns probabilities to an integer number of occurrences. It has long been used as an alternative to the binomial distribution when the sample size is large and the occurrence of the event of interest is rare (Haight, 1967). The Poisson distribution describes a wide range of phenomena in the sciences such as the number of pollen grains collected in regions of a sticky plate exposed to the open air and the number of white blood cells found in a cubic centimeter of blood. In many biological, epidemiological, and medical studies, the comparison of Poisson means (i.e., the average number of occurrences per unit of time or space) from two independent populations is of great research interest. For instance, in a breast cancer study two groups of women were compared to determine whether those who had been examined using X-ray fluoroscopy during treatment for tuberculosis had a higher rate of breast cancer than those who had not been examined using X-ray fluoroscopy (Rothman and Greenland, 1998; Graham et al., 2003). Forty-one cases of breast cancer in 28,010 person-years at risk are reported in the treatment group with women receiving X-ray fluoroscopy and 15 cases of breast cases in 19,017 person-years at risk in the control group with women not receiving X-ray fluoroscopy.

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The problem of comparing two Poisson means has long been studied in the literature. However, most of the studies focused on equal-size sampling frames. Given that the Poisson variates X_0 and X_1, Przyborowski and Wilenski (1940) first proposed a conditional test for the equality of two Poisson rates using the binomial distribution of the conditional distribution of X_1 given $X_0 + X_1$. Afterward, Chapman (1952), Birnbaum (1953), Brownlee (1967) and Gail (1974) studied other test statistics based on the conditional approach and their properties. On the other hand, Hald (1960), Ractliffe (1964), Cox and Lewis (1966), Haight (1967), Detre and White (1970) and Sichel (1973) investigated the use of asymptotic tests for the equality of two Poisson rates based on large-sample normal approximation.

Development for unequal-size sampling frames has received more attention in the last two decades. Shiue and Bain (1982) derived an uniformly most powerful unbiased (UMPU) test for the equality of two Poisson rates and showed that a test based on the normal approximation is nearly as powerful as the UMPU test. Huffman (1984) proposed an improved asymptotic test statistic which accelerates the rate of convergence to normality by a variance stabilizing transformation. After that, Thode (1997) considered an alternative test statistic which has higher power than the Shiue and Bain statistic. Krishnamoorthy and Thomson (2004) proposed a test based on an estimated p-value and compared it with the conditional test. They showed that their test is more powerful than the conditional test. Recently, Ng and Tang (2005) studied testing the equality of two Poisson means over unequal-size sampling frames.

Nevertheless, there are limitations in the current literature and little comparison among existing procedures especially for situations with unequal-size sample frames. Most of the works in the literature are concerned about the test of equality of two Poisson rates (difference between two Poisson rates equal to zero). Systematic comparisons among different test procedures for testing non-zero difference between two Poisson means under unequal-size sampling frames have not been conducted.

In this article we consider the problem of comparing two Poisson means over unequal-size sampling frames. In Section 2, we consider three different kinds of test procedures, (a) asymptotic tests, (b) tests based on approximate p-value method and (c) the likelihood ratio test. The first two kinds of test procedures are based on test statistics that differ in their estimates of the variance of the difference of the Poisson rates. In Section 3, we conduct a Monte Carlo simulation study to investigate the performance of the various methods in terms of significance level (i.e., type I error rate or size) and power. Simulation results are presented in Section 4 with discussion. Some recommendations under different settings are also discussed. In Section 5, we demonstrate our methodologies with a real example from a breast cancer study.

2. Test procedures

Suppose that two independent Poisson processes (with parameters λ_0 and λ_1) are observed for fixed (known) sampling frames t_0 and t_1, respectively, and let X_0 and X_1 be the corresponding number of outcomes. That is, $X_i \sim$ Poisson (λ_i) with $\lambda_i = t_i\gamma_i$ for $i = 0, 1$. Here, γ_i is the event rate or the incidence rate. We denote the observed values of X_0 and X_1 by x_0 and x_1, respectively. We wish to test the following one-sided hypotheses

$$\begin{aligned} \mathrm{H}_0 &: \gamma_0 - \gamma_1 = \delta, \\ \text{against } \mathrm{H}_1 &: \gamma_0 - \gamma_1 > \delta, \end{aligned} \tag{1}$$

where $\delta \geqslant 0$ is a pre-specified number.

In this article, we consider the following estimates for the unknown parameters γ_i $(i = 0, 1)$:

- *Unconstrained maximum likelihood estimate*: The maximum likelihood estimate of γ_i is given by

$$\hat{\gamma}_i = \frac{X_i}{t_i}, \quad i = 0, 1. \tag{2}$$

- *Unbiased estimate*: Krishnamoorthy and Thomson (2004) proposed the following unbiased estimates for γ_0 and γ_1 under the null hypothesis $\mathrm{H}_0 : \gamma_0 - \gamma_1 = \delta$:

$$\tilde{\gamma}_0 = \frac{X_0 + X_1}{t_0 + t_1} + \frac{\delta t_1}{t_0 + t_1}$$