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# Distance magic graphs $G \times C_n$

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A B S T R A C T

A $\Gamma$-distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection $f$ from $V$ to an Abelian group $\Gamma$ of order $n$ such that the weight $w(x) = \sum_{y \in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*.

In this paper we will show that if $G$ is a graph of order $n = 2^p(2k+1)$ for some natural numbers $p, k$ such that $\deg(v) \equiv c \pmod{2^{p+2}}$ for some constant $c$ for any $v \in V(G)$, then there exists a $\Gamma$-distance magic labeling for any Abelian group $\Gamma$ of order $4n$ for the direct product $G \times C_4$. Moreover if $c$ is even, then there exists a $\Gamma$-distance magic labeling for any Abelian group $\Gamma$ of order $8n$ for the direct product $G \times C_8$.



## 1. Introduction and preliminaries

All graphs considered in this paper are simple finite graphs. We use $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph $G$. The *neighborhood* $N(x)$ or more precisely $N_G(x)$, when needed, of a vertex $x$ is the set of vertices adjacent to $x$, and the *degree* $d(x)$ of $x$ is $|N(x)|$, the size of the neighborhood of $x$. By $C_n$ we denote a cycle on $n$ vertices.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order $n$ is a bijection $l: V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer $\mu$ (called the *magic constant*) such that $\sum_{y \in N_G(x)} l(y) = \mu$ for every $x \in V$. If a graph $G$ admits a distance magic labeling, then we say that $G$ is a *distance magic graph* [4]. The sum $\sum_{y \in N_G(x)} l(y)$ is called the *weight of the vertex* $x$ and is denoted by $w(x)$.

The concept of distance magic labeling has been motivated by the construction of magic squares. It is worth mentioning that finding an $r$-regular distance magic graph turns out equivalent to finding equalized incomplete tournament EIT$(n, r)$ [10]. In an *equalized incomplete tournament* EIT$(n, r)$ of $n$ teams with $r$ rounds, every team plays exactly $r$ other teams and the total strength of the opponents that team $i$ plays is $k$. Thus, it is easy to notice that finding an EIT$(n, r)$ is the same as finding a distance magic labeling of any $r$-regular graph on $n$ vertices. For a survey, we refer the reader to [4,11].

The following observations were independently proved:

**Observation 1.1** ([17,19,20,22]). *Let $G$ be an $r$-regular distance magic graph on $n$ vertices. Then $\mu = \frac{r(n+1)}{2}$.*

**Observation 1.2** ([17,19,20,22]). *No $r$-regular graph with $r$ odd can be a distance magic graph.*

We recall three out of four standard graph products (see [12,15]). All three, the *Cartesian product* $G \Box H$, *lexicographic product* $G \circ H$ and the *direct product* $G \times H$, are graphs with the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in:

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- $G\square H$ if and only if $g = g'$ and $h$ is adjacent to $h'$ in $H$, or $h = h'$ and $g$ is adjacent to $g'$ in $G$;
- $G \circ H$ if and only if either $g$ is adjacent to $g'$ in $G$ or $g = g'$ and $h$ is adjacent to $h'$ in $H$;
- $G \times H$ if $g$ is adjacent to $g'$ in $G$ and $h$ is adjacent to $h'$ in $H$.

The graph $G \circ H$ is also called the *composition* and is denoted by $G[H]$ (see [13]). The product $G \times H$, which is also known as *Kronecker product*, *tensor product*, *categorical product* and *graph conjunction*, is the most natural graph product. The direct product is commutative, associative, and it has several applications, for instance it may be used as a model for concurrency in multiprocessor systems [18]. Some other applications can be found in [16].

Some graphs which are distance magic among (some) products can be seen in [1–3,5,6,8,9,19,21].

**Theorem 1.3** (*[19]*)**.** *Let $r \geq 1$, $n \geq 3$, $G$ be an $r$-regular graph and $C_n$ be the cycle of length $n$. The graph $G \circ C_n$ admits a distance magic labeling if and only if $n = 4$.*

**Theorem 1.4** (*[19]*)**.** *Let $G$ be an arbitrary regular graph. Then $G \circ \overline{K}_n$ is distance magic for any even $n$.*

**Theorem 1.5** (*[2]*)**.** *Let $G$ be an arbitrary regular graph. Then $G \times C_4$ is distance magic.*

Since the Cartesian product $C_n\square C_m$ is distance magic if and only if $n = m \equiv 2 \pmod 4$ (see [21]), the graph $C_n\square C_4$ is not distance magic for any $n$, we state the problem:

**Problem 1.6.** If $G$ is a regular graph, determine if there is a distance magic labeling of $G\square C_4$.

The following problem was posed in [4].

**Problem 1.7** (*[4]*)**.** If $G$ is a non-regular graph, determine if there is a distance magic labeling of $G \circ C_4$.

A similar problem for the direct product was stated in [8]:

**Problem 1.8** (*[8]*)**.** If $G$ is a non-regular graph, determine if there is a distance magic labeling of $G \times C_4$.

Moreover it was proved that:

**Theorem 1.9** (*[8]*)**.** *Let $m$ and $n$ be two positive integers such that $m \leq n$. The graph $K_{m,n} \times C_4$ is a distance magic graph if and only if the following conditions hold:*

1. $m + n \equiv 0 \pmod 2$ *and*
2. $1 = 2(8n + 1)^2 - (8m + 8n + 1)^2$ *or* $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2}-1}{8}$.

Froncek in [9] defined the notion of *group distance magic graphs*, i.e. the graphs allowing a bijective labeling of vertices with elements of an Abelian group resulting in constant sums of neighbor labels.

**Definition 1.10.** A $\Gamma$-distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection $f$ from $V$ to an Abelian group $\Gamma$ of order $n$ such that the weight $w(x) = \sum_{y\in N_G(x)} f(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the magic constant. A graph $G$ is called a *group distance magic graph* if there exists a $\Gamma$-distance magic labeling for every Abelian group $\Gamma$ of order $|V(G)|$.

The connection between distance magic graphs and $\Gamma$-distance magic graphs is as follows. Let $G$ be a distance magic graph of order $n$ with the magic constant $\mu'$. If we replace the label $n$ in a distance magic labeling for the graph $G$ by the label 0, then we obtain a $\mathbb{Z}_n$-distance magic labeling for the graph $G$ with the magic constant $\mu \equiv \mu' \pmod n$. Hence every distance magic graph with $n$ vertices admits a $\mathbb{Z}_n$-distance magic labeling. Although a $\mathbb{Z}_n$-distance magic graph on $n$ vertices is not necessarily a distance magic graph (see [9]), it was proved that Observation 1.2 also holds for a $\mathbb{Z}_n$-distance magic labeling [7].

**Observation 1.11** (*[7]*)**.** *Let $r$ be a positive odd integer. No $r$-regular graph on $n$ vertices can be a $\mathbb{Z}_n$-distance magic graph.*

The following theorem was proved in [9]:

**Theorem 1.12** (*[9]*)**.** *The Cartesian product $C_m\square C_k$, $m, k \geq 3$, is a $\mathbb{Z}_{mk}$-distance magic graph if and only if $km$ is even.*

Froncek also showed that the graph $C_{2k}\square C_{2k}$ has a $\mathbb{Z}_2^{2k}$-distance magic labeling for $k \geq 2$ and $\mu = (0, 0, \ldots, 0)$ [9].
Cichacz proved:

**Theorem 1.13** (*[6]*)**.** *Let $G$ be a graph of order $n$ and $\Gamma$ be an Abelian group of order $4n$. If $n = 2^p(2k + 1)$ for some natural numbers $p$, $k$ and $\deg(v) \equiv c \pmod{2^{p+1}}$ for some constant $c$ for any $v \in V(G)$, then there exists a $\Gamma$-distance magic labeling for the graph $G \circ C_4$.*

It seems that the direct product is the natural choice among (standard) products to deal with $\Gamma$-distance magic graphs and group distance magic graphs in general. The reason for this is that the direct product is a suitable product if we observe graphs as categories. Hence it should perform well with the product of (Abelian) groups, as illustrated in the following theorem.