Discrete Applied Mathematics 177 (2014) 95–100

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Ore's condition for completely independent spanning trees[☆]

Genghua Fan [a], Yanmei Hong [b], Qinghai Liu [a,*]

[a] Center for Discrete Mathematics, Fuzhou University, Fuzhou, Fujian, 350002, China

[b] College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350108, China

ARTICLE INFO

Article history:
Received 14 January 2014
Received in revised form 2 June 2014
Accepted 7 June 2014
Available online 3 July 2014

Keywords:
Completely independent spanning trees (CIST)
Line graphs

ABSTRACT

Two edge-disjoint spanning trees of a graph G are completely independent if the two paths connecting any two vertices of G in the two trees are internally disjoint. It has been asked whether sufficient conditions for hamiltonian graphs are also sufficient for the existence of two completely independent spanning trees (CISTs). We prove that it is true for the classical Ore-condition. That is, if G is a graph on n vertices in which each pair of non-adjacent vertices have degree-sum at least n, then G has two CISTs. It is known that the line graph of every 4-edge connected graph is hamiltonian. We prove that this is also true for CISTs: the line graph of every 4-edge connected graph has two CISTs. Thomassen conjectured that every 4-connected line graph is hamiltonian. Unfortunately, being 4-connected is not enough for the existence of two CISTs in line graphs. We prove that there are infinitely many 4-connected line graphs that do not have two CISTs.

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. $G - U$ is the subgraph induced by $V(G) \setminus U$. $\partial_G(U)$ is the set of edges with exactly one end in U. $N_G(U)$ is the set of vertices in $V(G) \setminus U$ which are adjacent to vertices in U. For a vertex $u \in V(G)$, we write $G - u$, $\partial_G(u)$ and $N_G(u)$ instead of $G - \{u\}$, $\partial_G(\{u\})$ and $N_G(\{u\})$ respectively. $d_G(u) = |\partial_G(u)|$ is called the *degree* of u. $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ is called the *minimum degree* of G. The *order* of a graph is the number of vertices it contains.

For two vertices x, y in a graph G, an (x, y)-path is a path connecting x and y. Two (x, y)-paths P_1 and P_2 are *openly disjoint* if they have no edge in common and no vertex in common except for x, y. Let $T_1, T_2, \ldots, T_k$ be k spanning trees in G. If for any distinct vertices x, y in G, the (x, y)-paths in $T_1, T_2, \ldots, T_k$ are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are *completely independent spanning trees* (CISTs) in G.

By the definition, completely independent spanning trees are also edge-disjoint spanning trees, and thus can be applied to the communication problems to which edge-disjoint spanning trees are applied. As mentioned by Hasunuma [6], in applications to fault-tolerant broadcasting in parallel computers, if G has k completely independent spanning trees, then deleting any $k - 1$ vertices from G still results in a connected graph, and therefore, completely independent spanning trees can tolerate not only edge-faults but also vertex-faults. From an algorithmic point of view, Hasunuma [5] showed that the problem of finding two completely independent spanning trees in a given graph is NP-hard. Completely independent

☆ This research is supported by NSFC (11301086, 11326214).

* Corresponding author.
E-mail addresses: fan@fzu.edu.cn (G. Fan), yhong@fzu.edu.cn (Y. Hong), qliu@fzu.edu.cn (Q. Liu).

http://dx.doi.org/10.1016/j.dam.2014.06.002
0166-218X/

spanning trees can be viewed as independent spanning trees rooted at any vertex. Readers are referred to [7] for independent spanning trees and its applications.

It is well known [8,12] that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Motivated by this, Hasunuma [5] conjectured that every $2k$-connected graph has k completely independent spanning trees. However, Péterfalvi [10] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two completely independent spanning trees. In [4], Hasunuma study the completely independent spanning trees in some graphs. Recently, Araki [1] gave a Dirac's condition characterization for the existence of completely independent spanning trees. Let G be a graph and V_1, V_2 be two disjoint sets of vertices in G. Denote by $B(V_1, V_2)$ the bipartite graph induced by the edges with one end in V_1 and the other end in V_2. A partition (V_1, V_2) of $V(G)$ is called a *CIST*-partition if both $G[V_1]$ and $G[V_2]$ are connected and no component of $B(V_1, V_2)$ is a tree. In [1], Araki proved the following theorem.

Theorem 1.1 (*Araki [1]*)**.** *A connected graph G has two CISTs if and only if G has a CIST-partition.*

In the same paper, Araki [1] obtained some interesting results about the existence of two CISTs and showed that some sufficient conditions for a graph to be hamiltonian are also sufficient for the existence of two CISTs. A classical result on hamiltonian graphs is the Dirac's Theorem [2] that a graph G on n vertices is hamiltonian if $d_G(v) \geq n/2$ for each $v \in V(G)$. Another well-known result on hamiltonian graphs is the one by Fleischner [3] that the square of every 2-connected graph is hamiltonian. Araki [1] proved that both conditions are also sufficient for the existence of two CISTs.

Theorem 1.2 (*Araki [1]*)**.** *A graph on $n \geq 7$ vertices has two CISTs if $\delta(G) \geq n/2$.*

Theorem 1.3 (*Araki [1]*)**.** *The square of every 2-connected graph has two CISTs.*

With these results, Araki [1] asked whether other known conditions for a graph to be hamiltonian can also imply the existence of two CISTs. This paper gives some answers to this question. A well-known such condition is the Ore-condition [9] that a graph G on n vertices is hamiltonian if $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$. In the next section, we prove that the Ore-condition is sufficient for the existence of two CISTs.

Theorem 1.4. *Let G be a graph on n vertices, $n > 8$. If $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u, v, then G has two completely independent spanning trees.*

The line graph $L(G)$ of a graph G is the graph in which $V(L(G)) = E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding two edges are incident with the same vertex in G. It is known (see [13]) that the line graph of every 4-edge connected graph is hamiltonian. At the end of the paper, we prove that this condition for hamiltonian cycles in line graphs also guarantees the existence of two CISTs.

Theorem 1.5. *The line graph of every 4-edge connected graph has two completely independent spanning trees.*

Thomassen [11] conjectured that every 4-connected line graph is hamiltonian. Is the condition of being 4-connected sufficient for line graphs to have two CISTs? Unfortunately, the answer is negative. The following theorem will be proved in Section 3.

Theorem 1.6. *There are infinitely many 4-connected line graphs that do not have two completely independent spanning trees.*

2. Proof of Theorem 1.4

Suppose, to the contrary, that G does not have two CISTs. Thus by Theorem 1.1, G has no CIST-partition. For simplicity, let $\delta = \delta(G)$. By Theorem 1.2, $\delta < n/2$. The following claim is obvious from the assumption of the theorem.

Claim 1. *For any two non-adjacent vertices x, y, $|N_G(x) \cap N_G(y)| \geq 2$ with equality only if $N_G(x) \cup N_G(v) = V(G) \setminus \{x, y\}$.*

Claim 2. *G is 3-connected.*

Suppose, to the contrary, that $\{x, y\}$ is a cut of G and X, Y are two components of $G - \{x, y\}$. By Claim 1, for any $u \in X, v \in Y, N_G(u) = V(X) \cup \{x, y\} \setminus \{u\}$ and $N_G(v) = V(Y) \cup \{x, y\} \setminus \{v\}$ and $V(G) = V(X) \cup V(Y) \cup \{x, y\}$. By the arbitrariness of u and v, $V(X) \cup \{x, y\}$ and $V(Y) \cup \{x, y\}$ induce two complete graphs with at most one edge (xy) missing. As $n > 8$, one of X and Y has order at least 3, say X. Let $u_0, u_1, u_2 \in X$, $V_1 = \{u_0, x\}$ and $V_2 = V(G) \setminus V_1$. Then (V_1, V_2) is a partition of G and both $G[V_1]$ and $G[V_2]$ are connected. Also, every vertex in V_2 has at least one neighbor in V_1, and $xu_1u_0u_2x$ forms a 4-cycle in $B(V_1, V_2)$. Hence, (V_1, V_2) is a CIST-partition of G, a contradiction.

Claim 3. *For any three vertices x, y, z with $xz, yz \in E(G)$, $G - \{x, y, z\}$ is connected.*

Suppose, to the contrary, that x, y, z are such three vertices such that $G - \{x, y, z\}$ is disconnected. Let X be an arbitrary component of $G - \{x, y, z\}$, and Y be the union of other components.