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An efficient algorithm to determine all shortest paths in Sierpiński graphs[☆]

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ABSTRACT

In the *Switching Tower of Hanoi* interpretation of *Sierpiński graphs* S_p^n, the *P2 decision problem* is to find out whether the largest moving disc has to be transferred once or twice in a shortest path between two given states/vertices. We construct an essentially optimal algorithm thus extending Romik's approach for $p = 3$ to the general case. The algorithm makes use of three automata and the underlying theory includes a simple argument for the fact that there are at most two shortest paths between any two vertices. The total number of pairs leading to non-unique solutions is determined and employing a Markov chain argument it is shown that the number of input pairs needed for the decision is bounded above by a number independent of n. Elementary algorithms for the length of the shortest path(s) and the best first move/edge are also presented.

0. Introduction

Among the open problems in [6, Chapter 9] is the challenge to "design an automaton analogous to Romik's automaton for a 'P2 task' in S_p^n, $p \geq 4$". In fact, in [13] D. Romik presented an algorithm which determines the distance between any two vertices of the Sierpiński graph S_3^n by deciding about the number of necessary moves of the largest disc in an optimal solution of the corresponding task in the Switching Tower of Hanoi interpretation of that graph. This interpretation, together with a more general definition of *Sierpiński graphs* S_p^n, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, had been proposed in [8] by S. Klavžar and U. Milutinović. The common source of these questions was an open problem announced in [4, p. 179] concerning the *P2 decision problem* for the Tower of Hanoi, represented by the *Hanoi graph* H_3^n, which in turn is isomorphic to S_3^n; cf. the discussions in [6, Sections 2.4 and 4.1]. In [2], L.L. Cristea and B. Steinsky investigated the decision problem on another related class of graphs, the *Sierpiński triangle graphs* (cf. [9, p. 463]), on the *Sierpiński triangle* itself and on their 3-dimensional generalizations. Fundamental metric properties of Sierpiński graphs can be found in [12,7].

We will now solve the P2 decision problem for all S_p^n, thereby using, if not otherwise stated, definitions and notations from [6].[1] More specifically, we will construct an algorithm which, given vertices s and t of S_p^n (we also call (s, t) a *task*), returns the index of the largest disc (LD) moved on an optimal (i.e. minimal length) s, t-path and the answer to the decision question whether this disc moves once or twice or if both strategies lead to a shortest path. In the latter two cases, the

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1 Most notably, for $q \in \mathbb{N}_0$ we write $[q]$ and $[q]_0$ for the q-segments $\{1, \ldots, q\}$ and $\{0, \ldots, q-1\}$ of $\mathbb{N}$ and $\mathbb{N}_0$, respectively. Moreover, $\binom{K}{2}$ stands for the set of all subsets of size 2 of a set K.

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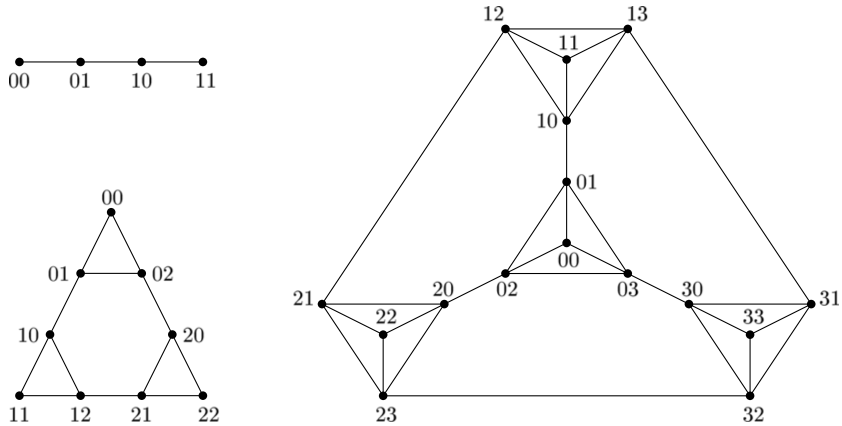

Fig. 1. The graphs S_2^2 (top left), S_3^2 (bottom left), and S_4^2 (right).

index of the subgraph different from initial and final subgraph (i.e. the *shortcut peg*) will also be detected. The goal of that algorithm is to obtain these output data with as little input as possible. With the information flowing from the algorithm it is then easy to determine the optimal edge from s, i.e. the *best first move* (BFM), and the distance $\mathrm{d}(s, t)$ of s and t in S_p^n, albeit using all information about these vertices. We begin in Section 1 with the theory, based on the methods in [8,13] and [6, Sections 2.4.3, 4.1 and 4.2.1], which will provide the correctness proof for the algorithm described in Section 2. In the concluding Section 3 we will analyse the complexity of the algorithm.

1. The theory

Just as the Tower of Hanoi game (cf. [6]), the *Switching Tower of Hanoi* is played with a certain number of moveable discs of different size stacked on some fixed vertical pegs. Any legal distribution of all discs among the pegs, i.e. with no larger disc lying on top of a smaller one, is called a (*regular*) *state*. Unlike in its famous archetype, where only one disc may be moved at a time, a move of the Switching Tower of Hanoi consists of the exchange of a topmost disc on one peg with the subtower of all smaller discs on top of another peg, including the case where the single disc is the smallest one and the corresponding subtower therefore empty. The P2 task is then, starting from a given arbitrary state, to find a (shortest) sequence of moves to get to another prescribed state.

As shown in [8, Theorem 1], the corresponding state graph, whose vertices are the states and the edges of which stand for the moves, is isomorphic to the *Sierpiński graph* S_p^n, where the *base* $p \in \mathbb{N}$ represents the number of pegs and the *exponent* $n \in \mathbb{N}_0$ is the number of discs. The graph is defined by

$$V(S_p^n) = [p]_0^n \quad \text{and} \quad E(S_p^n) = \left\{ \{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\} \mid \{i, j\} \in \binom{[p]_0}{2},\ d \in [n],\ \underline{s} \in [p]_0^{n-d} \right\};$$

here a vertex $s \in [p]_0^n$ is written in the form $s_n \ldots s_1$ and describes a state where $s_d \in [p]_0$ is the label of the peg the disc $d \in [n]$ is lying on.[2] An edge can be viewed as the switch of disc d on peg i with the tower of $d - 1$ smaller discs on peg j while the positions of larger discs, collected in $\underline{s}$, remain unchanged. For some examples of Sierpiński graphs, see Fig. 1.

An equivalent recursive definition [6, (4.7)] shows that the graphs S_p^n are connected; the canonical distance function will be denoted by d. The most important observation is[3]

$$\mathrm{d}(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) \cdot 2^{d-1}, \tag{0}$$

where j^n is the *extreme vertex* corresponding to the *perfect state* when all discs lie on peg j. This is Proposition 4.5 of [6], where the proof can be found. Moreover, the shortest path from s to j^n is unique; cf. also [8, Lemma 4]. Uniqueness may be lost, however, for $p \geq 3$ and $n \geq 2$ if both initial state s and goal state t are arbitrary. (The case $p = 1$ or $n = 0$ is trivial because there is only one vertex and no edge, for $p = 2$ we have $S_2^n \cong P_{2^n}$, the path graph on 2^n vertices (cf. Fig. 1), and finally $S_p^1 \cong K_p$, the complete graph of order p, for any p.) A simplest example is with $s = 01$ and $t = 21$ in S_3^2, where two shortest paths 01, 02, 20, 21 and 01, 10, 12, 21 of length 3 differ by the number of moves of the LD, called LDMs, namely 1 or 2 (see Fig. 1). It is also possible that 2 LDMs are *necessary* on a shortest path as in the task (011, 211) in S_3^3. It will turn out that this is already the worst that can happen! (This is in strong contrast to the situation in general *Hanoi graphs* H_p^n, where up to

[2] Discs are numbered according to increasing size.

[3] We employ *Iverson's convention* that $(\mathfrak{S}) = 1$, if statement $\mathfrak{S}$ is true, and $(\mathfrak{S}) = 0$, if $\mathfrak{S}$ is false.