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On the local spectra of the subconstituents of a vertex set and completely pseudo-regular codes[☆]

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ABSTRACT

In this paper we study the relation between the local spectrum of a vertex set C and the local spectra of its subconstituents. In particular, it is shown that, when C is a completely regular code, such spectra are uniquely determined by the local spectra of C. Moreover, we obtain a new characterization for completely pseudo-regular codes, and consequently for completely regular codes, in terms of the relation between the local spectrum of an extremal set of vertices and the local spectrum of its antipodal set. We also present a new proof of the version of the spectral excess theorem for extremal sets of vertices.

1. Introduction

The notion of local spectrum (see Section 2) was first introduced by Fiol, Garriga, and Yebra [14] for a single vertex of a graph. In that paper, such a concept was used to obtain several quasi-spectral characterizations of local (pseudo)-distance-regularity. In the study of pseudo-distance-regularity around a set of vertices, first done by Fiol and Garriga [12], which particularizes to that of completely regular codes when the graph is regular, the local spectrum is generalized to a set of vertices C (and, thus, it is called the C-local spectrum). As commented in the same paper, when we study the graph from a 'base' vertex subset C, its local spectrum plays a role similar to the one played by the (standard) spectrum for studying the whole graph.

In this work we are interested in the study of the relation between the local spectrum of a vertex set and the local spectra of the elements of the distance partition associated with it (also known as its subconstituents). Thus, Section 2 is devoted to define the local spectrum of a subset of vertices in a graph. In Section 3 completely pseudo-regular codes are introduced and we discuss some known results of special interest. Our main results can be found in Sections 4 and 5, where we give sufficient conditions implying a tight relation between the local spectrum of a set of vertices and that of each of its subconstituents. As a consequence, we obtain a new characterization of completely (pseudo-)regular codes. In the way, we also obtain some information about the structure of the local spectrum of the subconstituents associated with a completely pseudo-regular code and we give a new proof of a result by Fiol and Garriga [12,13], which can be seen as the spectral excess theorem, due to the same authors [11], for sets of vertices.

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Before going into our study, let us first give some notation. In this paper $\Gamma = (V, E)$ stands for a simple connected graph with a vertex set $V = \{1, 2, \ldots, n\}$. Each vertex $i \in V$ is identified with the i-th unit coordinate (column) vector $\boldsymbol{e}_i$ and $\mathcal{V} \cong \mathbb{R}^n$ denotes the vector space of formal linear combinations of its vertices. The distance function in Γ is denoted by $\operatorname{dist}(\cdot, \cdot)$. Given a set of vertices $C \subset V$, the distance from a vertex i to C is given by the expression $\operatorname{dist}(i, C) = \min\{\operatorname{dist}(i, j) | j \in C\}$. We denote by $\varepsilon_C = \max_{i\in V} \operatorname{dist}(i, C)$ the eccentricity of C. Notice that, in the context of coding theory, the parameter ε_C corresponds to the covering radius of the code C.

As usual, $\boldsymbol{A}$ stands for the adjacency matrix of Γ, with a set of different eigenvalues $\operatorname{ev}\Gamma = \operatorname{ev}\boldsymbol{A} = \{\lambda_0, \lambda_1, \ldots, \lambda_d\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. The spectrum of Γ is

$$\operatorname{sp}\Gamma = \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\},$$

where $m(\lambda_l)$ is the multiplicity of the eigenvalue λ_l. We denote by $\mathcal{E}_l = \ker(\boldsymbol{A} - \lambda_l \boldsymbol{I})$ the eigenspace of $\boldsymbol{A}$ corresponding to λ_l. Recall that, since Γ is connected, $\mathcal{E}_0$ is one-dimensional, and all its elements are eigenvectors having all its components either positive or negative (see, for instance, Biggs [1] or Cvetković, Doob, and Sachs [6]). Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathcal{E}_0$ the unique positive eigenvector of Γ with minimum component equal to 1.

Note that $\mathcal{V}$ is a module over the quotient ring $\mathbb{R}[x]/(Z)$, where (Z) is the ideal generated by the minimal polynomial of $\boldsymbol{A}$, $Z = \prod_{l=0}^{d}(x - \lambda_l)$, with product defined by

$$p\boldsymbol{u} = p(\boldsymbol{A})\boldsymbol{u} \quad \text{for every } p \in \mathbb{R}[x]/(Z) \text{ and } \boldsymbol{u} \in \mathcal{V}.$$

With this notation, let us remark that the orthogonal projection $\boldsymbol{E}_l$ of $\mathcal{V}$ onto the eigenspace $\mathcal{E}_l$ corresponds to

$$\boldsymbol{E}_l\boldsymbol{u} = Z_l\boldsymbol{u}, \quad \boldsymbol{u} \in \mathcal{V},$$

where Z_l, $l = 0, 1, \ldots, d$, is the Lagrange interpolating polynomial satisfying $Z_l(\lambda_h) = \delta_{lh}$, that is,

$$Z_l = \frac{(-1)^l}{\pi_l} \prod_{0\le h\le d,\, h\ne l} (x - \lambda_h),$$

with π_l being the moment-like parameter given by $\pi_l = \prod_{0\le h\le d,\, h\ne l} |\lambda_l - \lambda_h|$.

2. Local spectrum of a vertex set

Given a set of vertices C of Γ, define the map $\boldsymbol{\rho} : \mathcal{P}(V) \mapsto \mathcal{V}$ by $\boldsymbol{\rho}\emptyset = \boldsymbol{0}$ and $\boldsymbol{\rho}C = \sum_{i\in C} \nu_i \boldsymbol{e}_i$ for $C \ne \emptyset$. Consider the spectral decomposition of the unit vector $\boldsymbol{e}_C = \boldsymbol{\rho}C/\|\boldsymbol{\rho}C\| = \boldsymbol{z}_C(\lambda_0) + \boldsymbol{z}_C(\lambda_1) + \cdots + \boldsymbol{z}_C(\lambda_d)$, that is $\boldsymbol{z}_C(\lambda_l) = \boldsymbol{E}_l\boldsymbol{e}_C \in \mathcal{E}_l$, $0 \le l \le d$. The C-*local multiplicity* of the eigenvalue λ_l is defined by

$$m_C(\lambda_l) = \langle \boldsymbol{E}_l\boldsymbol{e}_C, \boldsymbol{e}_C\rangle = \|\boldsymbol{z}_C(\lambda_l)\|^2.$$

If $\mu_0 > \mu_1 > \cdots > \mu_{d_C}$ are the eigenvalues of Γ with nonzero C-local multiplicity, the C-*local spectrum* of Γ is defined by

$$\operatorname{sp}_C\Gamma = \{\mu_0^{m_C(\mu_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\},$$

and we denote by $\operatorname{ev}_C\Gamma = \{\mu_0, \mu_1, \ldots, \mu_{d_C}\}$, $\mu_0 > \mu_1 > \cdots > \mu_{d_C}$, the set of different eigenvalues in the C-local spectrum. Let us remark that, as $\boldsymbol{E}_0\boldsymbol{e}_C = \frac{\langle \boldsymbol{e}_C, \boldsymbol{\nu}\rangle}{\|\boldsymbol{\nu}\|^2}\boldsymbol{\nu} = \frac{\|\boldsymbol{\rho}C\|}{\|\boldsymbol{\nu}\|^2}\boldsymbol{\nu}$, we have $m_C(\lambda_0) = \frac{\|\boldsymbol{\rho}C\|^2}{\|\boldsymbol{\nu}\|^2} \ne 0$, and hence $\mu_0 = \lambda_0$. The parameter d_C is called the *dual degree* of C and it provides an upper bound for the eccentricity of the vertex set, $\varepsilon_C \le d_C$ (see [12]). When the equality is attained we say that C is *extremal*.

Notice that, if Γ is regular, then $\boldsymbol{\nu} = \boldsymbol{j}$ is the all-1 vector, and $\boldsymbol{\rho}C$ is the characteristic vector of C, $\boldsymbol{\chi} = \boldsymbol{\chi}_C$. Then, the above concepts reduce to standard Delsarte's theory [7,8] (see also Brouwer, Cohen and Neumaier [2, Section 2.5]). For instance, in this context, the sequence of C-local multiplicities $m_C(\lambda_0), \ldots, m_C(\lambda_d)$ is the so-called *MacWilliams transform* of the vector $\boldsymbol{e}_C$.

Consider the idempotents $\boldsymbol{E}_l^C$, $0 \le l \le d_C$, corresponding to the members of the C-local spectrum, that is $\boldsymbol{E}_l^C$ is the projection of $\mathcal{V}_C = \bigoplus_{\lambda_h\in \operatorname{ev}_C\Gamma} \mathcal{E}_h$ onto the eigenspace corresponding to μ_l. As we have done for the standard spectrum of the graph, we define for each $\mu_l \in \operatorname{ev}_C\Gamma$ the moment-like parameter $\pi_l(C) = \prod_{0\le h\le d_C,\, h\ne l} |\mu_l - \mu_h|$ and consider the Lagrange interpolation polynomial

$$Z_l^C = \frac{(-1)^l}{\pi_l(C)} \prod_{0\le h\le d_C,\, h\ne l} (x - \mu_h) \tag{1}$$

which gives $Z_l^C(\boldsymbol{A}) = \boldsymbol{E}_l^C$.

3. Completely pseudo-regular codes

In this section we review some known results on completely pseudo-regular codes. These results are formulated in terms of C-local pseudo-distance-regularity, which extends the notion of local distance-regularity from single vertices to subsets