Discrete Applied Mathematics 176 (2014) 19–23

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Spectral characterizations of almost complete graphs

Marc Cámara, Willem H. Haemers *

Tilburg University, The Netherlands

ARTICLE INFO

Article history:
Received 20 November 2012
Received in revised form 18 June 2013
Accepted 11 August 2013
Available online 6 September 2013

Keywords:
Graph spectrum
Spectral characterization
Cospectral graphs

ABSTRACT

We investigate when a complete graph K_n with some edges deleted is determined by its adjacency spectrum. It is shown to be the case if the deleted edges form a matching, a complete graph K_m provided $m \leq n-2$, or a complete bipartite graph. If the edges of a path are deleted we prove that the graph is determined by its generalized spectrum (that is, the spectrum together with the spectrum of the complement). When at most five edges are deleted from K_n, there is just one pair of nonisomorphic cospectral graphs. We construct nonisomorphic cospectral graphs (with cospectral complements) for all n if six or more edges are deleted from K_n, provided that n is big enough.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

Two graphs for which the adjacency matrices have the same spectrum are called *cospectral*. A graph G is *determined by its spectrum* (*DS* for short) if every graph cospectral with G is isomorphic with G. Spectral characterizations of graphs (with respect to various matrices) did get much attention in the recent past; see [4,5]. It has been conjectured by the second author that almost all graphs are DS. Truth of this conjecture would mean that the spectrum gives a useful fingerprint for a graph. The paradox is that it is difficult to prove that a given graph is DS. Not very many classes of graphs are known to be DS. These include for example the path P_n, the cycle C_n and the complete graph K_n. A number of papers have appeared that prove spectral characterizations for more complicated cases. Very often such graphs have relatively few edges, like T-shape trees and lollipop graphs (see [11,1]). Not many results are known if G has many edges, that is, the complement of G has few edges. If G is regular, or if one considers the spectrum of the Laplacian matrix, then a graph is DS if and only if the complement is. However, with respect to the adjacency spectrum of a nonregular graph G with few edges, the characterization problem for the complement of G is most of the time much harder than for G. For example for the path P_n, there is a straightforward proof that P_n is DS (see for example [4]). However, for the complement of P_n the proof is rather involved (see [7]).

If H is a subgraph of a graph G, then the graph obtained from G by deleting the edges of H is denoted by $G \backslash H$. In this paper the following graphs are proved to be DS: $K_n \backslash \ell K_2$, $K_n \backslash K_m$ (provided $m \leq n-2$), $K_n \backslash K_{\ell,m}$ and $K_n \backslash G$, when G has at most four edges. We show that there is exactly one pair of nonisomorphic cospectral graphs if five edges are deleted from K_n. If six or more edges are deleted from K_n, one can obtain cospectral graphs for every n which is big enough.

The graph $\Gamma = P_\ell + (n-\ell)K_1$ (that is, Γ is the disjoint union of the path P_ℓ and $n-\ell$ isolated vertices) has a nonisomorphic cospectral mate if ℓ is odd and $5 \leq \ell \leq n-1$ (see [3]). The complements of these cospectral graphs are not cospectral. More generally, we prove that Γ is determined by the generalized spectrum (which is the spectrum of Γ together with the spectrum of the complement).

* Corresponding author. Tel.: +31 134662777.
E-mail address: haemers@uvt.nl (W.H. Haemers).

0166-218X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.08.002

2. Removing a matching or a complete graph

It is known that the adjacency spectrum determines the number of closed walks of any given length ℓ. For $\ell = 0, 2$ and 3 this implies that cospectral graphs have the same number of vertices, edges and triangles, respectively. Deleting one edge from K_n destroys $n-2$ triangles, and deleting m edges destroys at most $m(n-2)$ triangles, with equality if and only if the deleted edges form a matching. Therefore any graph with n vertices, $\binom{n}{2} - m$ edges and $\binom{m}{3} - m(n-2)$ triangles is $K_n \backslash mK_2$. Thus we can conclude the following proposition.

Proposition 2.1. *A graph obtained from K_n by removing the edges of a matching is DS.*

Suppose $G = K_n \backslash K_m$. Then G has adjacency matrix

$$A = \begin{bmatrix} O_m & J \\ J & J - I_{n-m} \end{bmatrix}$$

(as usual, O, J and I are the all-zero, all-one and identity matrix, respectively; indices indicate the order). We see that $\operatorname{rank} A = n - m + 1$ and $\operatorname{rank}(A + I) = m + 1$; hence, A has an eigenvalue 0 with multiplicity $m - 1$ and an eigenvalue -1 with multiplicity $n - m - 1$ (the remaining two eigenvalues of A are $\frac{1}{2}(n - m - 1 \pm \sqrt{n^2 - 3m^2 + 2mn - 2n + 2m + 1})$). It will be proved that G is DS provided that $m \leq n - 2$. The first step is a result of Smith [10].

Lemma 2.1. *If a graph G has only one positive eigenvalue, then G is a complete multipartite graph, possibly extended with some isolated vertices.*

Proof. Suppose that G has $F = K_2 + K_1$ as an induced subgraph. Let x be the isolated vertex in F. Assume that x is not isolated in G, then x is adjacent to some vertex y of G outside F. The vertices of F together with y induce a subgraph of G on four vertices containing two disjoint edges. There are just three such graphs: $2K_2$, P_4, and a triangle with one pendant edge. All three have a positive second largest eigenvalue which contradicts the interlacing inequalities (see [2, p. 19]). Therefore F is not an induced subgraph of G, and therefore any two nonadjacent vertices of G have the same neighbors, which proves the claim. □

Theorem 2.1. *If $m \leq n - 2$, then $K_n \backslash K_m$ is DS.*

Proof. Let G be a graph cospectral with $K_n \backslash K_m$. By the above lemma, G consist of a complete multipartite graph G' and possibly some isolated vertices. The two smallest eigenvalues of the complete tripartite graph $K_{2,2,1}$ are -2 and $1 - \sqrt{5}$. Both of these values are less than -1, and eigenvalue interlacing implies that $K_{2,2,1}$ is not an induced subgraph of G'. A complete multipartite graph not containing $K_{2,2,1}$ as an induced subgraph has at most two classes, or has at most one class with more than one vertex. Therefore G' is a complete bipartite graph or $G' = K_{n'} \backslash K_{m'}$ where $m' \leq n' - 2$. In the first case $G' = K_2$, or G' has no eigenvalue -1, so $n - m - 1 = 0$ which was excluded. In the second case, the eigenvalue -1 has multiplicity $n - m - 1$ in G and multiplicity $n' - m' - 1$ in G', so $n - m = n' - m'$. Moreover, G and G' have the same number of edges; hence, $(n - m)(n - 1)/2 + m(n - m) = (n' - m')(n' - 1)/2 + m'(n' - m')$. Therefore $G = G' = K_n \backslash K_m$. □

Note that if $m = n - 1$ the result need not be true. Then $K_n \backslash K_m = K_{1,n-1}$, and if ℓ divides $n - 1$, then $K_{1,n-1}$ is cospectral with $K_{\ell,k} + (n - \ell - k)K_1$, where $k = (n - 1)/\ell$.

3. The multiplicity of −1

The complete graph K_n has an eigenvalue -1 with multiplicity $n - 1$. If a few edges are deleted from K_n, then there will still be an eigenvalue -1 with large multiplicity. In this section we deal with graphs having the eigenvalue -1 with multiplicity at least $n - 3$. Clearly K_n is the only graph for which the multiplicity of -1 is $n - 1$.

Proposition 3.1. *Let G be a graph on n vertices having an eigenvalue -1 with multiplicity $n - 2$. Then G is the disjoint union of two complete graphs, and therefore G is DS.*

Proof. Suppose that G has eigenvalue -1 with multiplicity $n - 2$. Then $A + I$ has rank 2. Since $G \neq K_n$, we may assume that the first two rows of $A+I$ correspond to nonadjacent vertices. Clearly these rows are independent and, since $\operatorname{rank}(A+I) = 2$, all rows of $A + I$ are linear combination of the first two rows. Then it follows straightforwardly that G is the disjoint union of two complete graphs. Clearly, the spectrum determines the order of each of the complete graphs. □

The following theorem is an unpublished result by Van Dam, Haemers and Stevanović.

Theorem 3.1. *Let G be a graph with n vertices having an eigenvalue -1 with multiplicity $n - 3$. Then $G = K_n \backslash K_{\ell,m}$, where $\ell, m \geq 1$, $\ell + m \leq n - 1$, or $G = K_k + K_\ell + K_m$, where $k, \ell, m \geq 1$, $k + \ell + m = n$.*

Download English Version:

https://daneshyari.com/en/article/418308

Download Persian Version:

https://daneshyari.com/article/418308

Daneshyari.com