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The spectrum and toughness of regular graphs

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ABSTRACT

In 1995, Brouwer proved that the toughness of a connected k-regular graph G is at least $k/\lambda - 2$, where λ is the maximum absolute value of the non-trivial eigenvalues of G. Brouwer conjectured that one can improve this lower bound to $k/\lambda - 1$ and that many graphs (especially graphs attaining equality in the Hoffman ratio bound for the independence number) have toughness equal to k/λ. In this paper, we improve Brouwer's spectral bound when the toughness is small and we determine the exact value of the toughness for many strongly regular graphs attaining equality in the Hoffman ratio bound such as Lattice graphs, Triangular graphs, complements of Triangular graphs and complements of point-graphs of generalized quadrangles. For all these graphs with the exception of the Petersen graph, we confirm Brouwer's intuition by showing that the toughness equals $k/(-\lambda_{min})$, where λ_{min} is the smallest eigenvalue of the adjacency matrix of the graph.

1. Introduction

The toughness $t(G)$ of a connected graph G is the minimum of $\frac{|S|}{c(G\backslash S)}$, where the minimum is taken over all vertex subsets S whose removal disconnects G, and $c(G\backslash S)$ denotes the number of components of the graph obtained by removing the vertices of S from G. A graph G is called t-tough if $t(G) \geq t$. Chvátal [10] introduced this parameter in 1973 to capture combinatorial properties related to the cycle structure of a graph. The toughness of a graph is related to many other important properties of a graph such as Hamiltonicity, and the existence of various factors, cycles or spanning trees and it is a hard parameter to determine exactly (see the survey [2]). Two of Chvátal conjectures from [10] motivated a lot of subsequent work. The first conjecture stated that there exists some $t_0 > 0$ such that any graph with toughness greater than t_0 is Hamiltonian. This conjecture is open at present time and Bauer, Broersma and Veldman [3] showed that if such a t_0 exists, then it must be at least 9/4. The second conjecture of Chvátal asserted the existence of $t_1 > 0$ such that any graph with toughness greater than t_1 is pancyclic. This was disproved by several authors including Alon [1], who showed that there are graphs of arbitrarily large girth and toughness. Alon's results relied heavily on the following theorem relating the toughness of a regular graph and its eigenvalues. If G is a connected k-regular graph on n vertices, we denote the eigenvalues of its adjacency matrix as follows: $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and we let $\lambda = \max(|\lambda_2|, |\lambda_n|)$.

Theorem 1.1 (*Alon [1]*). *If G is a connected k-regular graph, then*

$$t(G) > \frac{1}{3}\left(\frac{k^2}{k\lambda + \lambda^2} - 1\right). \qquad (1)$$

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Around the same time and independently, Brouwer [6] discovered slightly better relations between the toughness of a regular graph and its eigenvalues.

Theorem 1.2 (*Brouwer [6]*)**.** *If G is a connected k-regular graph, then*

$$t(G) > \frac{k}{\lambda} - 2. \tag{2}$$

Brouwer [7] conjectured that the lower bound of the previous theorem can be improved to $t(G) \geq \frac{k}{\lambda} - 1$ for any connected k-regular graph G. This bound would be best possible as there exists regular bipartite graphs with toughness very close to 0. Brouwer [7] mentioned the existence of such graphs, but did not provide any explicit examples. At the suggestion of one of the referees, we briefly describe a construction of such graphs here. Take k disjoint copies of the bipartite complete graph $K_{k,k}$ without one edge, add two new vertices and make each of these new vertices adjacent to one vertex of degree $k-1$ in each copy of $K_{k,k}$ minus one edge. The resulting graph is bipartite k-regular and has toughness at most $2/k$ since deleting the two new vertices creates k components.

Liu and Chen [15] found some relations between the Laplacian eigenvalues and the toughness of a graph. They also improved the eigenvalue conditions of Alon and Brouwer for guaranteeing 1-toughness.

Theorem 1.3 (*Liu and Chen [15]*)**.** *If G is a connected k-regular graph and*

$$\lambda_2 < \begin{cases} k - 1 + \dfrac{3}{k+1}, & k \text{ even} \\ k - 1 + \dfrac{2}{k+1}, & k \text{ odd} \end{cases} \tag{3}$$

then $t(G) \geq 1$.

In the first part of our paper, we improve Theorems 1.1–1.3 in certain cases. For small τ, we obtain a better eigenvalue condition than Alon's or Brouwer's that implies a regular graph is τ-tough. We also determine a best possible sufficient eigenvalue condition for a regular graph to be 1-tough improving the above result of Liu and Chen. We note here that Bauer, van den Heuvel, Morgana and Schmeichel [4,5] proved that recognizing 1-tough graphs is an NP-hard problem for regular graphs of valency at least 3. Our improvements are the following two results.

Theorem 1.4. *Let G be a connected k-regular graph on n vertices,* $k \geq 3$*, with adjacency eigenvalues* $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ *and edge-connectivity* κ'*. If* $\tau \leq \kappa'/k$ *is a positive number such that* $\lambda_2(G) < k - \frac{\tau k}{k+1}$*, then* $t(G) \geq \tau$.

Theorem 1.5. *If G is a connected k-regular graph and*

$$\lambda_2(G) < \begin{cases} \dfrac{k - 2 + \sqrt{k^2 + 8}}{2} & \text{when } k \text{ is odd} \\ \dfrac{k - 2 + \sqrt{k^2 + 12}}{2} & \text{when } k \text{ is even} \end{cases} \tag{4}$$

then $t(G) \geq 1$.

The proofs of Theorems 1.4 and 1.5 are similar to the one of Liu and Chen [15] and are contained in Section 2. We show that Theorem 1.5 is best possible in the sense that for each $k \geq 3$, we construct examples of k-regular graphs whose second largest eigenvalue equals the right hand-side of inequality (4), but whose toughness is less than 1. These examples are described in Section 3. Our examples are regular graphs of diameter 4 and their existence also answers a question of Liu and Chen [15, p. 1088] about the minimum possible diameter of a regular graph with toughness less than 1.

In [7], Brouwer also stated that he believed that $t(G) = \frac{k}{\lambda}$ for many graphs G. Brouwer's reasoning hinged on the fact that a connected k-regular graph G with n vertices attaining equality in the Hoffman ratio bound (meaning that the independence number $\alpha(G)$ of G equals $\frac{n(-\lambda_{\min})}{k-\lambda_{\min}}$; see e.g. [8, Chapter 3] or [13, Chapter 9]) and having $\lambda = -\lambda_{\min}$, is likely to have toughness equal to $k/\lambda = k/(-\lambda_{\min})$. Brouwer deduced that for such a graph G, k/λ is definitely an upper bound for the toughness, (as one can take S to be the complement of an independent set of maximum size $\frac{n\lambda}{k-\lambda}$ and then $t(G) \leq \frac{|S|}{c(G\backslash S)} = k/\lambda$) and suggested that for many such graphs k/λ is the exact value of $t(G)$.

In the second part of the paper, we determine the exact value of the toughness of several families of strongly regular graphs attaining equality in Hoffman ratio bound, namely the Lattice graphs, the Triangular graphs, the complements of the Triangular graphs and the complements of the point-graphs of generalized quadrangles. Moreover, for each graph G above, we determine the disconnecting sets of vertices S such that $\frac{|S|}{c(G\backslash S)}$ equals the toughness of G. We show that for all these graphs except the Petersen graph, the toughness equals $k/(-\lambda_{\min})$, where k is the degree of regularity and $\lambda_{\min}$ is the smallest eigenvalue of the adjacency matrix. These results are contained in Section 4. In Section 4.1, we prove that the