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Fat Hoffman graphs with smallest eigenvalue greater than −3

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ABSTRACT

In this paper, we give a combinatorial characterization of the special graphs of fat Hoffman graphs containing $\mathfrak{K}_{1,2}$ with smallest eigenvalue greater than −3, where $\mathfrak{K}_{1,2}$ is the Hoffman graph having one slim vertex and two fat vertices.

1. Introduction

In the field of Spectral Graph Theory, one of the important research problems is to characterize graphs with bounded smallest eigenvalue. In 1976, using root systems, P.J. Cameron, J.M. Goethals, J.J. Seidel, and E.E. Shult [2] characterized graphs whose adjacency matrices have smallest eigenvalue at least −2. Their results revealed that graphs with smallest eigenvalue at least −2 are generalized line graphs, except a finite number of graphs represented by the root system E_8. We refer the reader to the monograph [3], for a complete account of this theory. In 1977, A.J. Hoffman [8] studied graphs whose adjacency matrices have smallest eigenvalue at least $-1-\sqrt{2}$ by using a technique of adding cliques to graphs. In 1995, R. Woo and A. Neumaier [14] formulated Hoffman's idea by introducing the notion of Hoffman graphs and generalizations of line graphs. Hoffman graphs were subsequently studied in [9,11–13,15]. In particular, H.J. Jang, J. Koolen, A. Munemasa, and T. Taniguchi [9] proposed a scheme to classify fat indecomposable Hoffman graphs with smallest eigenvalue at least −3. The present paper completes a partial case of this scheme. While there are quite a few Hoffman graphs with smallest eigenvalue at most −3 (see Section 3), there are strong restrictions on Hoffman graphs with smallest eigenvalue greater than −3. The counterpart of this problem for ordinary graphs is the classification of graphs with smallest eigenvalue greater than −2, given by [4]. In this paper, we consider fat indecomposable Hoffman graphs with smallest eigenvalue greater than −3, under the additional assumption that there exists a slim vertex with two fat neighbors.

Our result can also be regarded as a reformulation of a classical result of Hoffman [7] in terms of Hoffman graphs. Let $\hat{A}(G, v^*)$ denote the adjacency matrix of a graph G, modified by putting −1 in the diagonal position corresponding to a vertex v^*. Hoffman [7, Lemma 2.1] has shown that $\hat{A}(L(T), e)$ has smallest eigenvalue greater than −2 whenever e is an end edge of a tree T, where $L(T)$ denotes the line graph of T. Moreover, under a conjecturally redundant assumption, Hoffman [7, Lemma 2.2] has shown that the smallest eigenvalue of $\hat{A}(L(T), e)$ is a limit point of the set of smallest eigenvalues of

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graphs. Denoting by $\lambda_{\min}(A)$ the smallest eigenvalue of a real symmetric matrix A, this implies that $\lambda_{\min}(\hat{A}(L(T), e)) - 1$ is also a limit point of the set of smallest eigenvalues of graphs. In fact, $\lambda_{\min}(\hat{A}(L(T), e)) - 1$ can be regarded as the smallest eigenvalue of a Hoffman graph, and by Hoffman's limit theorem [8, Proposition 3.1] (see also [6,9]), it is a limit point of the set of smallest eigenvalues of graphs. In this way, we obtain limit points of smallest eigenvalues of graphs with smallest eigenvalue greater than -3.

The goal of this paper is to characterize the special graphs of fat indecomposable Hoffman graphs with smallest eigenvalue greater than -3 containing a slim vertex having two fat neighbors. As a consequence, we show in Theorem 5.2 that, if the smallest eigenvalue of $\hat{A}(G, v^*)$ is greater than -2, then G is the line graph of a tree T and v^* corresponds to an end edge of T.

The organization of the paper is as follows. In Section 2, we give basic results on Hoffman graphs and block graphs which are needed in later sections. In Section 3, we show that various Hoffman graphs have smallest eigenvalue at most -3. These graphs will play a role of forbidden subgraphs for the family of fat Hoffman graphs with smallest eigenvalue greater than -3. In Section 4, we give our main theorem which characterizes the special graphs of fat indecomposable Hoffman graphs with smallest eigenvalue greater than -3 containing a slim vertex having two fat neighbors. Finally, in Section 5, we give an extension of a lemma of Hoffman [7] about the smallest eigenvalue of the modified adjacency matrix of a graph.

2. Preliminaries

2.1. Hoffman graphs

A *Hoffman graph* $\mathfrak{H}$ is a pair of a (simple undirected) graph $(V(\mathfrak{H}), E(\mathfrak{H}))$ and a distinguished coclique $F \subseteq V(\mathfrak{H})$. A vertex in F is called a *fat vertex* and a vertex in $V(\mathfrak{H}) \setminus F$ is called a *slim vertex*. We denote F and $V(\mathfrak{H}) \setminus F$ by $V^s(\mathfrak{H})$ and $V^f(\mathfrak{H})$, respectively. In this paper, we assume that no fat vertex is isolated.

For a vertex x of a Hoffman graph $\mathfrak{H}$, a *slim neighbor* (resp. a *fat neighbor*) of x in $\mathfrak{H}$ is a slim vertex (resp. a fat vertex) y of $\mathfrak{H}$ such that $\{x, y\}$ is an edge of $\mathfrak{H}$. We denote by the set of slim neighbors (resp. fat neighbors) of x in $\mathfrak{H}$ by $N^s_{\mathfrak{H}}(x)$ (resp. $N^f_{\mathfrak{H}}(x)$). A Hoffman graph $\mathfrak{H}$ is said to be *fat* if every slim vertex of $\mathfrak{H}$ has a fat neighbor, and $\mathfrak{H}$ is said to be *slim* if $\mathfrak{H}$ has no fat vertex.

Two Hoffman graphs $\mathfrak{H}$ and $\mathfrak{H}'$ are said to be *isomorphic* if there exists a bijection $\phi : V(\mathfrak{H}) \to V(\mathfrak{H}')$ such that $\phi(V^s(\mathfrak{H})) = V^s(\mathfrak{H}')$, $\phi(V^f(\mathfrak{H})) = V^f(\mathfrak{H}')$, and $\{x, y\} \in E(\mathfrak{H})$ if and only if $\{\phi(x), \phi(y)\} \in E(\mathfrak{H})$. A Hoffman graph $\mathfrak{H}'$ is called an *induced Hoffman subgraph* of a Hoffman graph $\mathfrak{H}$ if $V^s(\mathfrak{H}') \subseteq V^s(\mathfrak{H})$, $V^f(\mathfrak{H}') \subseteq V^f(\mathfrak{H})$, and $E(\mathfrak{H}') = \{\{x, y\} \in E(\mathfrak{H}) \mid x, y \in V(\mathfrak{H}')\}$.

Let

$$A(\mathfrak{H}) = \begin{pmatrix} A^s(\mathfrak{H}) & C(\mathfrak{H}) \\ C(\mathfrak{H})^T & O \end{pmatrix}$$

be the adjacency matrix of a Hoffman graph $\mathfrak{H}$, in a labeling in which the slim vertices come first and the fat vertices come last. The *eigenvalues* of $\mathfrak{H}$ are defined to be the eigenvalues of the real symmetric matrix

$$B(\mathfrak{H}) = A^s(\mathfrak{H}) - C(\mathfrak{H})C(\mathfrak{H})^T.$$

We denote the smallest eigenvalue of $B(\mathfrak{H})$ by $\lambda_{\min}(\mathfrak{H})$.

Lemma 2.1 *([14, Corollary 3.3]). If $\mathfrak{H}'$ is an induced Hoffman subgraph of a Hoffman graph $\mathfrak{H}$, then $\lambda_{\min}(\mathfrak{H}') \geq \lambda_{\min}(\mathfrak{H})$ holds.*

A *decomposition* of a Hoffman graph $\mathfrak{H}$ is a family $\{\mathfrak{H}^i\}_{i=1}^n$ of non-empty induced Hoffman subgraphs of $\mathfrak{H}$ satisfying the following conditions:

(i) $V(\mathfrak{H}) = \bigcup_{i=1}^n V(\mathfrak{H}^i)$;
(ii) $V^s(\mathfrak{H}^i) \cap V^s(\mathfrak{H}^j) = \emptyset$ if $i \neq j$;
(iii) For each $x \in V^s(\mathfrak{H}^i)$, $N^f_{\mathfrak{H}}(x) \subseteq V^f(\mathfrak{H}^i)$
(iv) If $x \in V^s(\mathfrak{H}^i)$, $y \in V^s(\mathfrak{H}^j)$, and $i \neq j$, then $|N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| \leq 1$, and $|N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| = 1$ if and only if $\{x, y\} \in E(\mathfrak{H})$.

A Hoffman graph $\mathfrak{H}$ is said to be *decomposable* if $\mathfrak{H}$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$ with $n \geq 2$, and $\mathfrak{H}$ is said to be *indecomposable* if $\mathfrak{H}$ is not decomposable.

Lemma 2.2 *([9, Lemma 2.12]). If a Hoffman graph $\mathfrak{H}$ has a decomposition $\{\mathfrak{H}^i\}_{i=1}^n$, then $\lambda_{\min}(\mathfrak{H}) = \min\{\lambda_{\min}(\mathfrak{H}^i) \mid 1 \leq i \leq n\}$.*

Let $\mathfrak{H}$ be a Hoffman graph and let m and N be positive integers. A *reduced representation of norm m* of $\mathfrak{H}$ is a map $\psi : V^s(\mathfrak{H}) \to \mathbb{R}^N$ such that

$$\langle \psi(x), \psi(y) \rangle = \begin{cases} m - |N^f_{\mathfrak{H}}(x)| & \text{if } x = y, \\ 1 - |N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| & \text{if } \{x, y\} \in E(\mathfrak{H}), \\ -|N^f_{\mathfrak{H}}(x) \cap N^f_{\mathfrak{H}}(y)| & \text{otherwise}, \end{cases}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^N$. It follows immediately from the definitions that $B(\mathfrak{H})_{x,y} = \langle \psi(x), \psi(y) \rangle$ holds for any distinct slim vertices x, y.