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Matrix power inequalities and the number of walks in graphs

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ABSTRACT

We unify and generalize several inequalities for the number w_k of walks of length k in graphs, and for the entry sum of matrix powers.

First, we present a weighted sandwich theorem for Hermitian matrices which generalizes a matrix theorem by Marcus and Newman and which further generalizes our former unification of inequalities for the number of walks in undirected graphs by Lagarias et al. and by Dress and Gutman. The new inequality uses an arbitrary nonnegative weighting of the indices (vertices) which allows to apply the theorem to index (vertex) subsets (i.e., inequalities considering the number $w_k(S, S)$ of walks of length k that start at a vertex of a given vertex subset S and that end within the same subset). We also deduce a stronger variation of the sandwich theorem for the case of positive-semidefinite Hermitian matrices which generalizes another inequality of Marcus and Newman.

Further, we show a similar theorem for nonnegative symmetric matrices which is another unification and generalization of inequalities for walk numbers by Erdős and Simonovits, by Dress and Gutman, and by Ilić and Stevanović.

In the last part, we generalize lower bounds for the spectral radius of adjacency matrices and upper bounds for the energy of graphs.

1. Introduction

1.1. Motivation

In this work, we investigate powers of Hermitian matrices. We present inequalities relating entries of different powers of a matrix to each other. In the special case of an adjacency matrix, the entries of its kth power are the numbers of walks of length k between the vertices that correspond to the row/column indices. Similar to the number of present edges in a (sub)graph that is used to define the (statistical) density of this (sub)graph by dividing it through the number of possible edges in a complete graph on the same vertex set, the number of walks of length k induces a density of order k: the ratio of the number of k-walks to the maximum possible number of k-walks [20]. Applying our inequalities to this concept of density yields statements about the relation between densities of different orders.

Another application of our results is found in symmetric models of computation, which exhibit undirected configuration graphs. One particular example for such a model is the symmetric Turing machine which was defined by Lewis and Papadimitriou [23] to characterize the complexity class Symmetric Logspace (*SL*) for which undirected s, t-connectivity (USTCON) is a complete problem. In this context, the number of computation paths consisting of k transitions equals the number of walks of length k in the corresponding configuration graph starting at the initial configuration. Assuming that the configuration graph is finite, it is also interesting to investigate the total number of different computation path segments of certain lengths starting at arbitrary vertices. Bounds could be given in terms of the number of configurations, total number of transitions,

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number of transitions incident to each configuration, and so on. Other bounds could take into account the number of computation path segments of other lengths.

A more universal application of counting the number of walks is to exploit their relationship to the largest eigenvalue λ_1 of adjacency matrices. To this end, we derive new lower bounds for λ_1 in terms of the number of walks. In turn, λ_1 can be used to bound other important graph measures. In [14], Hoffman obtained the bound $1 - \lambda_1/\lambda_n \leq \chi$ for the chromatic number χ, relating it to the ratio of λ_1 to the smallest eigenvalue λ_n. Also, the clique number ω can be bounded using Wilf's inequality [34] $n/(n-\lambda_1) \leq \omega$. Another interesting application of λ_1 considers the SIS model of disease spreading, in which a susceptible (S) individual is possibly infected by an already infected (I) neighbor, and subsequently may become cured again. If an infected individual infects a certain neighbor with probability β and is cured with probability δ, then the expected size of the infected part of the population reduces exponentially if $\beta/\delta < 1/\lambda_1$, i.e., $1/\lambda_1$ is the epidemic threshold in this model (see [10,3]). Besides the spreading of viruses in biological and computer networks, this model can also be applied to rumor spreading and information broadcasting.

More information on applications of graph spectra can be found in [27,6,4,33].

1.2. Notation and basic facts

Throughout the paper, we assume that $\mathbb{N}$ denotes the set of nonnegative integers and that $[n]$ is the set $\{1, \ldots, n\}$. Let A be an $n \times n$-matrix with complex entries. We write $\mathrm{sum}(A)$ for the sum of the entries of A. For the kth power A^k of A, we use $a_{i,j}^{[k]}$ to denote the (i, j)-entry of A^k and define $a_{i,j} = a_{i,j}^{[1]}$ for convenience.

Let $G = (V, E)$ be an undirected graph having n vertices, m edges, and adjacency matrix A. We investigate directed walks, i.e., sequences of vertices, where each pair of consecutive vertices is connected by an edge. Nodes and edges can be used repeatedly in the same walk. The length k of a walk is counted in terms of edges. For $k \in \mathbb{N}$ and $x, y \in V$, let $w_k(x, y)$ denote the number of walks of length k that start at vertex x and end at vertex y. Since G is undirected, we have $w_k(x, y) = w_k(y, x)$. For vertex subsets $X, Y \subseteq V$, $w_k(X, Y)$ denotes the number of walks of length k starting at a vertex of X and ending at a vertex of Y. We write $w_k(x) = \sum_{y \in V} w_k(x, y)$ for the number of walks of length k that start at node x (which is the same as the number of walks of length k that end at node x). Accordingly, $w_k = \sum_{x \in V} w_k(x)$ denotes the total number of walks of length k. For the adjacency matrix A of a graph G, we will frequently make use of the equalities $w_k = \mathrm{sum}(A^k)$ and $w_k(i, j) = a_{i,j}^{[k]}$.

1.3. Related work

1.3.1. Inequalities for the number of walks

First, we briefly review results for undirected graphs. Let $a, b, c, k, \ell, p \in \mathbb{N}$ be nonnegative integers. Erdős and Simonovits (and actually Godsil) [9] noticed that the following inequality using the average degree $\overline{d} = 2m/n$ can be shown using results of Mulholland and Smith [28,29], Blakley and Roy [2], and London [24]:

$$w_k \geq n\overline{d}^k = n\left(\frac{w_1}{w_0}\right)^k \quad \text{or} \quad w_1^k \leq w_0^{k-1} w_k. \tag{1}$$

Lagarias, Mazo, Shepp, and McKay [21,22] showed that

$$w_{2a+b} \cdot w_b \leq w_0 \cdot w_{2(a+b)}, \tag{2}$$

and presented counterexamples for $w_r \cdot w_s \leq n \cdot w_{r+s}$ whenever $r + s$ is odd and $r, s \geq 1$. Dress and Gutman [8] reported the inequality

$$w_{a+b}^2 \leq w_{2a} \cdot w_{2b}. \tag{3}$$

These inequalities were generalized by Täubig et al. [32] to the "sandwich theorem" (for nonnegative integers $a, b, c \in \mathbb{N}$):

$$w_{2a+c} \cdot w_{2a+2b+c} \leq w_{2a} \cdot w_{2(a+b+c)} \tag{4}$$

and the following inequality (for nonnegative integers $k, \ell, p \in \mathbb{N}$ and $k \geq 2$ or $w_{2\ell} > 0$):

$$w_{2\ell+p}^k \leq w_{2\ell}^{k-1} \cdot w_{2\ell+pk}. \tag{5}$$

For all graphs with $w_{2\ell} > 0$ (i.e., for graphs with at least one edge or for $\ell = 0$), this is equivalent to $(w_{2\ell+p}/w_{2\ell})^k \leq w_{2\ell+pk}/w_{2\ell}$ and $(w_{2\ell+p}/w_{2\ell})^{k-1} \leq w_{2\ell+pk}/w_{2\ell+p}$.

They also showed that similar inequalities are valid for closed walks (for all $v \in V$):

$$w_{2a+c}(v, v) \cdot w_{2a+2b+c}(v, v) \leq w_{2a}(v, v) \cdot w_{2(a+b+c)}(v, v) \tag{6}$$

and, for $k \geq 2$ or $w_{2\ell}(v, v) > 0$,

$$w_{2\ell+p}(v, v)^k \leq w_{2\ell+pk}(v, v) \cdot w_{2\ell}(v, v)^{k-1}. \tag{7}$$