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Cayley graphs of diameter two and any degree with order half of the Moore bound

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ABSTRACT

It is well known that the number of vertices of a graph of diameter two and maximum degree d is at most $d^2 + 1$. The number $d^2 + 1$ is the Moore bound for diameter two. Let $C(d, 2)$ denote the largest order of a Cayley graph of degree d and diameter two. Up to now, the only known construction of Cayley graphs of diameter two valid for all degrees d is a construction giving $C(d, 2) > \frac{1}{4}d^2 + d$. However, there is a construction yielding Cayley graphs of diameter two, degree d and order $d^2 - O(d^{\frac{3}{2}})$ for an infinite set of degrees d of a special type.

In this article we present, for any integer $d \geq 4$, a construction of Cayley graphs of diameter two, degree d and of order $\frac{1}{2}d^2 - t$ for d even and of order $\frac{1}{2}(d^2 + d) - t$ for d odd, where $0 \leq t \leq 8$ is an integer depending on the congruence class of d modulo 8.

In addition, we show that, in asymptotic sense, the most of record Cayley graphs of diameter two is obtained by our construction.

1. Introduction

In graph theory, the *degree-diameter* problem is to find the largest order $n(d, k)$ of a graph with given maximum degree d and diameter k. There is a well known upper bound on the order $n(d, k)$ called the *Moore bound*, which gives for every $d, k \geq 1$ the bound $n(d, k) \leq 1 + d + d(d-1) + d(d-2)^2 + \cdots + d(d-1)^{k-1}$. The graphs satisfying the Moore bound are *Moore graphs*. If $k = 1$ or $d \leq 2$, the Moore bound is achieved by complete graphs K_{d+1} ($k = 1, d \geq 1$) and by odd-length cycles C_{2k+1} ($d = 2, k \geq 1$). For diameter $k = 2$ there are Moore graphs only for degrees $d = 2$ (the 5-cycle C_5), $d = 3$ (Petersen graph), $d = 7$ (Hoffman–Singleton graph) and (possibly) for $d = 57$ (a hypothetical graph of degree 57 on 3250 vertices). Note that if there is a Moore graph for $k = 2$ and $d = 57$, the graph is not vertex-transitive [2] and, in addition, it has very small automorphism group (of order at most 375) [5]. For other combinations of degrees and diameters there are no Moore graphs. A survey about the history and development on this topic can be found in [7].

For diameter $k = 2$ the Moore bound is $n(d, 2) \leq d^2 + 1$. It was shown in [3] that for every degrees $d \geq 4, d \neq 7, d \neq 57$ the bound on the order of a graph of diameter two and degree d is $n(d, 2) \leq d^2 - 1$. Brown's graphs [1] provide the lower bound on $n(d, 2)$ in the form $n(d, 2) \geq d^2 - d + 2$ if $(d-1)$ is a power of 2 and $n(d, 2) \geq d^2 - d + 1$ if $(d-1)$ is an odd prime power. In [10], the authors constructed modified Brown's graphs to show that $n(d, 2) \geq d^2 - d^{1.525}$ for all sufficiently large d.

It is known, that neither Brown's graphs nor their modification is vertex-transitive. Therefore, it is interesting to determine the maximum number of vertices of a vertex-transitive graph or a Cayley graph of degree d and diameter two. In

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what follows we will denote by $v(d, 2)$ and $C(d, 2)$ the maximum order of a vertex-transitive and Cayley graph, respectively, of diameter two and degree d. Since Cayley graphs are vertex transitive, we have $n(d, 2) \geq v(d, 2) \geq C(d, 2)$ for any degree d. In the case of vertex-transitive graphs, until recently the best lower bound was a construction in [6] giving vertex-transitive McKay–Miller–Širáň graphs of order $\frac{8}{9}(d + \frac{1}{2})^2$ for all degrees $d = \frac{1}{2}(3q - 1)$ where $q \equiv 1 \pmod 4$ is a prime power. It was shown in [6] that these graphs are non-Cayley.

For Cayley graphs, until now, the best known lower bound valid for all degrees d is a folklore bound $C(d, 2) \geq \lfloor \frac{d+2}{2} \rfloor \lceil \frac{d+2}{2} \rceil = \frac{1}{4}d^2 + d + 1$ for even d and $\frac{1}{4}d^2 + d + \frac{3}{4}$ for odd d. The groups used in this construction are direct products of cyclic groups $\mathcal{Z}_{\lfloor \frac{d+2}{2} \rfloor} \times \mathcal{Z}_{\lceil \frac{d+2}{2} \rceil}$. In [8], the authors constructed Cayley graphs of order $\frac{1}{2}(d + 1)^2$ for all degrees $d = 2q - 1$ where q is an odd prime power. Using non-trivial results on the distribution of primes, it has been shown in [10] that for all sufficiently large d, there is a Cayley graph of diameter two and of order $\frac{1}{2}d^2 - O(d^{1.525})$. Just very recently, in [9] the authors constructed Cayley graphs of order $d^2 - O(d^{\frac{3}{2}})$ for an infinite set of numbers of very special type.

In this article we show that for every integer $d \geq 4$ there is a Cayley graph of degree d, diameter two and of order $\frac{1}{2}d^2 - t$ for d even and of order $\frac{1}{2}(d^2 + d) - t$ for d odd, where $0 \leq t \leq 8$ is an integer depending on the congruence class of d modulo 8. This is a significant improvement of the known results valid for all degrees. In addition, we show that our construction provides infinitely many largest known Cayley graphs of diameter two.

2. The results

Let Γ be a finite group and let X be a finite generating set of Γ such that X is unit-free and it is closed under taking inverses. That is, $1_\Gamma \notin X$ and for each $x \in X$ we have $x^{-1} \in X$. The Cayley graph for the *underlying* group Γ and the *generating* set X is the graph $G = Cay(\Gamma, X)$ with vertex set Γ and edges of the form $\{g, gx\}$, $g \in \Gamma$, $x \in X$. Because the sets $\{g, g \cdot x\}$ and $\{gx, gx \cdot x^{-1}\}$ are the same, the Cayley graph G is undirected. Since the mapping $\varphi_h : V(G) \to V(G),\ h \in \Gamma$ defined by $\varphi_h(g) = hg,\ g \in V(g)$ is an automorphism of G, Cayley graphs are automatically vertex-transitive.

The dihedral group D_n of order $2n$ is the group with standard presentation

$$D_n = \langle a, b | a^n = b^2 = 1, ba = a^{-1}b \rangle \tag{1}$$

and the multiplicative cyclic group $\mathcal{Z}_n$ of order n is the group

$$\mathcal{Z}_n = \langle A | A^n = 1 \rangle. \tag{2}$$

The direct product of groups D_m and $\mathcal{Z}_n$ is the group of order $2mn$ with presentation

$$\Gamma = D_m \times \mathcal{Z}_n = \langle a, b, A | a^m = b^2 = A^n, ba = a^{-1}b, Aa = aA, bA = Ab \rangle. \tag{3}$$

We will write the elements of Γ in the form $a^iA^jb^k$, $i \in \{0, 1, \ldots, m - 1\}$, $j \in \{0, 1, \ldots, n - 1\}$, $k \in \{0, 1\}$. Throughout the paper, the unit element of any group will be denoted by "1".

The following definition plays a key role in our construction of Cayley graphs of diameter two.

Definition 1. Let i be an integer. We define the function $\delta : \mathcal{Z} \to \mathcal{Z}$ by

$$\delta(i) = i \cdot (-1)^{\lfloor i/2 \rfloor}. \tag{4}$$

That is, $\delta(i) = i$ for $i \equiv 0, 1 \pmod 4$ and $\delta(i) = -i$ for $i \equiv 2, 3 \pmod 4$.

To simplifying the notation, we will frequently denote the set of first positive integers by symbol $[n]$, that is, $[n] = \{1, 2, \ldots, n\}$.

Since our construction of Cayley graphs depends on the parity of degree d, in what follows we will consider two cases: case (I) for even degrees and case (II) for odd degrees d.

Theorem 1 (*Main Theorem*)**.** *Let $d \geq 4$ be an integer and let $s \in \{-2, -1, 0, 1\}$. Then there exist a group Γ and a generating set X for Γ such that the Cayley graph $G = Cay(\Gamma, X)$ has diameter two, degree d and order*

(I) $|G| = \frac{1}{2}(d - 2s)(d + 2s)$, *for* $d \equiv 2s + 4 \pmod 8$ *and*
(II) $|G| = \frac{1}{2}(d - 2s - 1)(d + 2s + 2)$, *for* $d \equiv 2s + 5 \pmod 8$.

Before proving the Main theorem, we will prove the following two useful Lemmas which we will need later in the proof of Theorem 2.

Lemma 1. *Let $n \geq 1$ be an odd number and let*

(I) $Y_0 = \{1 + \delta(i), -1 + \delta(i) | i \in \{1, 2, \ldots, 2n\}\} \pmod{4n}$
(II) $Y_1 = \{1 + \delta(i), -1 + \delta(i) | i \in \{1, 2, \ldots, 2n + 1\}\} \pmod{4n + 3}$.

Then

(I) $Y_0 = \{0, 2, 3, \ldots, 4n - 1\}$, *for* $n \neq 1$
$Y_0 = \{0, 1, 2, 3\}$, *for* $n = 1$
(II) $Y_1 = \{0, 2, 3, \ldots, 4n + 2\}$.