Discrete Applied Mathematics 173 (2014) 8–15

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Extremal problems related to Betti numbers of flag complexes ☆

Michał Adamaszek *

Fachbereich Mathematik, Universität Bremen, Bibliothekstr. 1, 28359 Bremen, Germany

ARTICLE INFO

Article history:
Received 17 April 2013
Received in revised form 1 April 2014
Accepted 2 April 2014
Available online 20 April 2014

Keywords:
Simplicial topology
Flag complex
Betti numbers
Independence complex
Neighbourhood complex

ABSTRACT

We study the problem of maximizing Betti numbers of simplicial complexes. We prove an upper bound of 1.32^n for the sum of Betti numbers of any n-vertex flag complex and 1.25^n for the independence complex of a triangle-free graph. These findings imply upper bounds for the Betti numbers of various related classes of spaces, including the neighbourhood complex of a graph. We also make some related observations.

1. Introduction

There is a number of topological and algebraic invariants associated to graphs and simplicial complexes. The starting point for this investigation is the following kind of extremal problem: given a bound on the size of the combinatorial input, how large values can those invariants take? For example, if a simplicial complex has at most n vertices, then it has at most 2^n faces, and trivially its homology groups can have total dimension at most 2^n. This is asymptotically optimal. The k-skeleton of the n-simplex is known to be homotopy equivalent to the wedge of $\binom{n}{k+1}$ spheres, hence for $k \approx n/2$ its homology has dimension approximately $\frac{2^n}{\sqrt{n}}$. This construction is optimal by [2].

Considerably better upper bounds can be obtained by considering restricted families of complexes. In this work we concentrate on flag complexes. A simplicial complex K is called *flag* if the set of faces of K coincides with the set of cliques in the 1-skeleton of K, hence K is also called a *clique complex*. Clique complexes appear throughout geometry, topology and combinatorics, some notable examples being order complexes of posets and Vietoris–Rips complexes of discrete metric spaces.

From the point of view of our applications it will be convenient to take the following, equivalent approach to flag complexes. The *independence complex* $\mathrm{Ind}(G)$ of a graph G is a simplicial complex whose faces are the independent sets of G (the sets $W \subseteq V(G)$ for which the induced subgraph $G[W]$ has no edges). Clearly the family of independence complexes of graphs is the same as the family of flag complexes as an independent set in G is a clique in the graph-theoretic complement $\overline{G}$.

We fix once and for all a field $\mathbf{k}$. For a finite type topological space X let $\mathrm{b}(X) = \sum_i \dim_{\mathbf{k}} \widetilde{H}_i(X; \mathbf{k})$ denote the *total Betti number* of X (we are using reduced homology; in particular $\mathrm{b}(\emptyset) = \dim_{\mathbf{k}} \widetilde{H}_{-1}(\emptyset; \mathbf{k}) = 1$). In Section 3 we will define

☆ This research was partially carried out when the author was a member of the Centre for Discrete Mathematics and its Applications (DIMAP) and the Mathematics Institute of the University of Warwick, Coventry, UK. Research supported by Engineering and Physical Sciences Research Council award EP/D063191/1 and a Deutsche Forschungsgemeinschaft (DFG) grant FE 1058/1-1.

* Tel.: +49 15143555486; fax: +49 4212184856.
E-mail address: aszek@mimuw.edu.pl.

http://dx.doi.org/10.1016/j.dam.2014.04.006
0166-218X/

Table 1
Lower and upper bounds appearing in this paper, with n denoting the number of vertices of G.

Quantity	Best construction	Upper bound
$\mathrm{b}(\mathrm{Ind}(G))$	(Example 6.1) 1.320^n	(Theorem 1.1) 1.320^n
$\mathrm{b}(\mathrm{Ind}(G))$, G triangle-free	(Example 6.2) 1.160^n	(Theorem 1.1) 1.250^n
$\mathrm{b}(\mathcal{N}(G))$	(Example 6.8) 1.316^n	(Theorem 1.2) 1.562^n
$\beta(G)$	(Example 6.5) 2.299^n	(Proposition 3.2) 2.320^n
$\beta(G)$, G triangle-free	(Example 6.6) 2.070^n	(Proposition 3.2) 2.250^n

constants

$$\Theta = 4^{1/5} \approx 1.32 \quad \text{and} \quad \Gamma \approx 1.25$$

for which we have the next result.

Theorem 1.1. *For any n-vertex graph G we have*

$$\mathrm{b}(\mathrm{Ind}(G)) \le \Theta^n.$$

If G is triangle-free then

$$\mathrm{b}(\mathrm{Ind}(G)) \le \Gamma^n.$$

Let us make a few comments. The first inequality can also be deduced from more general results of [12], although the above formulation seems far from being "well-known". For the smaller class of order complexes one can use an even earlier result of [20]. Theorem 1.1 provides a benchmark for studying families of graphs G for which $\mathrm{b}(\mathrm{Ind}(G))$ is exponential in the number of vertices n. Such graphs have appeared recently in mathematical physics, see for example [6,9,8]. Independence complexes of bipartite and triangle-free graphs have also received some attention [1,10], as well as those of other classes of graphs with forbidden induced subgraphs [5]. It should also be noted that the above are the extremal values, only approached by tailor-made constructions. The behaviour of Betti numbers for random or geometric flag complexes is a lot more tame, see [11,7].

The second part of Theorem 1.1 has various consequences listed below. Recall that the *neighbourhood complex* $\mathcal{N}(G)$ of a graph G is a simplicial complex whose vertices are the non-isolated vertices of G and whose faces are the vertex subsets which have a common neighbour in G. It is a well-known construction with classical applications to the theory of chromatic numbers of graphs [16].

Theorem 1.2. *We have the following upper bounds.*

(a) *If K is any simplicial complex with n vertices and m maximal faces then*

$$\mathrm{b}(K) \le \Gamma^{n+m}.$$

(b) *If G is a graph with n vertices then its neighbourhood complex $\mathcal{N}(G)$ satisfies*

$$\mathrm{b}(\mathcal{N}(G)) \le (\Gamma^2)^n.$$

(c) *If K is any simplicial complex with n vertices and m minimal non-faces then*

$$\mathrm{b}(K) \le \Gamma^{n+m}.$$

Clearly parts (a), (c) give useful bounds (better than 2^n) only when m is sufficiently small, approx. $m \le 2.1n$.

The paper is laid out as follows. Section 2 contains the necessary background and notation. In Section 3 we give a proof of Theorem 1.1 based on the inequality (2). As an immediate application we use Hochster's formula to derive an upper bound for the sum of algebraic Betti numbers $\beta(G)$ of flag complexes and edge ideals.

Theorem 1.2 is proved in Section 4 using the correspondence between arbitrary simplicial complexes and independence complexes of bipartite graphs.

In Section 5 we investigate complexes without missing d-faces, which are a generalization of flag complexes. We use Alexander duality to show an analogue, albeit much weaker, of Theorem 1.1 for such complexes. As an aside, we show how the same methods give bounds on the homological dimension in those classes of complexes.

Unfortunately, of all the upper bounds we present, only the first one in Theorem 1.1 is known to be tight. In Section 6 we will construct examples exhibiting the best lower bounds we were able to find. It is likely that some of them are in fact optimal. They are summarized in Table 1.

2. Preliminaries

We first introduce some notation and prove basic results.

Graphs. All graphs are finite, undirected and without multiple edges or loops. If $v \in V(G)$ then $N_G(v)$ is the set of neighbours of v and $N_G[v] = N_G(v) \cup \{v\}$. The degree of v is $\deg_G v = |N_G(v)|$ and $\mathrm{mindeg}(G)$ is the smallest degree of a vertex of G.