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On linear coloring of planar graphs with small girth ☆

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ABSTRACT

A vertex coloring of a graph G is linear if the subgraph induced by the vertices of any two color classes is the union of vertex-disjoint paths. In this paper, we study the linear coloring of graphs with small girth and prove that: (1) Every planar graph with maximum degree $\Delta \geq 39$ and girth $g \geq 6$ is linearly $(\lceil \frac{\Delta}{2} \rceil + 1)$-colorable. (2) There exists an integer Δ_0 such that every planar graph with maximum degree $\Delta \geq \Delta_0$ and girth $g \geq 5$ is linearly $(\lceil \frac{\Delta}{2} \rceil + 1)$-colorable. The latter result is best possible in some sense.

1. Introduction

Graphs considered in this paper are finite, simple and undirected. Unless stated otherwise, we follow the notations and terminology in [1]. For a planar graph G, we denote its *vertex set, edge set, face set, minimum degree* and *maximum degree* by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$, respectively. For a vertex v, $d(v)$ and $N(v)$ denote its degree and the set of its neighbors in G, respectively. For a face f of a planar graph G, we use $n_i(f)$ to denote the number of i-vertex incident with f. For integers k and d, a $k(d)$-*vertex* is a k-vertex adjacent to d 2-vertices. A k-*vertex* (or k-*face*) is a vertex (or a face) of degree k, a k^--*vertex* (or k^--*face*) is a vertex (or a face) of degree at most k, and a k^+-*vertex* (or k^+-*face*) is defined similarly.

A proper k-*coloring* of a graph G is a mapping c from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. Let c be a proper k-coloring of G. If the subgraph induced by the vertices of any two color classes with respect to c is the union of vertex-disjoint paths, then c is said to be *linear* and is called a *linear k-coloring* of G. The *linear chromatic number*, denoted by $lc(G)$, of the graph G is the least integer k such that G admits a linear k-coloring.

Yuster [11] first introduced the linear coloring of graphs. With the probabilistic method, he proved that $lc(G) = O(\Delta(G)^{\frac{3}{2}})$ for a general graph G and constructed graphs G with $lc(G) = \Omega(\Delta(G)^{\frac{3}{2}})$.

Another concept related to linear coloring is *frugal coloring* of graphs, considered by Hind et al. in [6]. In a coloring c of G, a vertex v of G is said to be k-*frugal* if every color appears at most $k-1$ times in the neighborhood of v. If every vertex is k-frugal in the coloring c, then G is said to be k-*frugal*, and the coloring c is called a k-frugal coloring of G. Obviously, a linear coloring is just a 3-frugal coloring. But the converse may not be true since in a 3-frugal coloring, bicolored cycles are permitted.

The upper bounds of linear chromatic number of graphs have been extensively investigated in the past years, especially for planar graphs. Let $g(G)$ (or g for simplicity) be the girth of a graph G, which is the length of a shortest cycle of G. Raspaud and Wang [8] proved that every planar graph G with girth g and maximum degree Δ has $lc(G) = \lceil \frac{\Delta}{2} \rceil + 1$ if G satisfies one

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of the following four conditions: (1) $g \geq 13$ and $\Delta \geq 3$; (2) $g \geq 11$ and $\Delta \geq 5$; (3) $g \geq 9$ and $\Delta \geq 7$; (4) $g \geq 7$ and $\Delta \geq 13$. lots of sufficient conditions for planar graphs and sparse graphs to have linear chromatic number close to the natural lower bound can be found in [2–5,7,9,10], interested readers are referred to them for more information on this topic.

It is obvious that $lc(G) \geq \lceil \frac{\Delta(G)}{2} \rceil + 1$. One interesting problem is to find the sufficient condition for graphs to attain this lower bound. The results in [8] show that planar graphs G can attain the lower bound if $g(G) \geq 7$ and $\Delta(G) \geq 13$.

In this paper, we consider the lower bound of linear coloring of planar graphs with small girth. Actually, we prove the following theorems.

Theorem 1.1. *Every planar graph with maximum degree $\Delta \geq 39$ and girth $g \geq 6$ is linearly $(\lceil \frac{\Delta}{2} \rceil + 1)$-colorable.*

Theorem 1.2. *There exists an integer Δ_0 such that every planar graph with maximum degree $\Delta \geq \Delta_0$ and girth $g \geq 5$ is linearly $(\lceil \frac{\Delta}{2} \rceil + 1)$-colorable.*

Theorem 1.2 is best possible in some sense since there exist planar graphs G with $g(G) = 4$ and arbitrary large maximum degree Δ such that $lc(G) \geq \lceil \frac{\Delta}{2} \rceil + 2$. Let us take $K_{2,n}$ for instance. It is easy to verify that $g(K_{2,n}) = 4$ and $K_{2,n}$ is planar. However, $lc(K_{2,n}) \geq \lceil \frac{\Delta}{2} \rceil + 2$ if $n \geq 2$.

Before starting our main work, we introduce some notations and terminology. Let c be a linear coloring of G, we use $c(v)$ to denote the color of a vertex v with respect to c. Similarly, for a vertex set $S \subseteq V(G)$, $c(S)$ is denoted to the set of colors assigned to the vertices of S with respect to c. We use $C_i(v)$ to denote the set of colors appears i times on the neighbors of v. Obviously, $i \in \{0, 1, 2\}$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. We prove this theorem by contradiction. Suppose the theorem is false. Let G be a minimal counterexample with the fewest vertices. Let $L = \{1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ be the color set. In the following, we first present some structures of G, then apply a discharging procedure to show that G does not exist. Hence, the theorem holds.

Lemma 2.1. $\delta(G) \geq 2$.

Lemma 2.2. *G does not contain a 2-vertex u with $N(u) = \{v, w\}$ such that $\lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil < \lceil \frac{\Delta}{2} \rceil + 1$.*

Proof. Suppose to the contrary, u is such a 2-vertex. By the choice of G, $G - u$ has a linear coloring c using $\lceil \frac{\Delta}{2} \rceil + 1$ colors. If $c(v) \neq c(w)$, then u can receive any color except for $c(v)$, $c(w)$ and those colors that appear twice on $N(v)$ or twice on $N(w)$. So the number of colors forbidden is at most $2 + \lfloor \frac{d(v)-1}{2} \rfloor + \lfloor \frac{d(w)-1}{2} \rfloor = \lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil$. Since $\lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil < \lceil \frac{\Delta}{2} \rceil + 1$, we can extend c to the whole graph G. Otherwise, if $c(v) = c(w)$, then at most $1 + |C_2(v) \cup C_2(w)| + |(C_1(v) \cap C_1(w))| \leq \lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil$ colors are forbidden to u. Hence, we can easily properly color u to obtain a linear coloring of G. A contradiction.

Lemma 2.3. *Let v be a 3-vertex of G with $N(v) = \{x, y, z\}$ such that $d(x) = 2$ and $d(y) \leq 3$. If $d(z) = 3$ with $N(z) = \{v, z_1, z_2\}$, then $\lfloor \frac{d(z_1)-1}{2} \rfloor + \lfloor \frac{d(z_2)-1}{2} \rfloor \geq \lceil \frac{\Delta}{2} \rceil - 3$.*

Proof. Suppose to the contrary. Let v be such a 3-vertex with $\lfloor \frac{d(z_1)-1}{2} \rfloor + \lfloor \frac{d(z_2)-1}{2} \rfloor \leq \lceil \frac{\Delta}{2} \rceil - 4$. Assume that u is the other neighbor of x. We consider the worst case that $d(u) = \Delta$ and $d(y) = 3$. Let $N(y) = \{v, y_1, y_2\}$. By the choice of G, $G-x$ admits a linear coloring c with $\lceil \frac{\Delta}{2} \rceil + 1$ colors. We will show that we can extend c to the whole graph G, which is a contradiction. We have the following possibilities.

Case 1. $c(u) = c(v)$ and $c(y) \neq c(z)$. If $|C_2(u)| = \lfloor \frac{\Delta-1}{2} \rfloor$ and Δ is odd, then $|C_0(u)| = 2$ and $|C_1(u)| = 0$. We can color x with $\alpha \notin \{c(u)\} \cup C_2(u)$. Since $c(y) \neq c(z)$, v is 3-frugal. Moreover $c(x) \notin C_2(u)$, no 2-colored cycle is induced. The obtained coloring is a linear coloring of G.

If $|C_2(u)| = \lfloor \frac{\Delta-1}{2} \rfloor$ and Δ is even, then $C_0(u) = \{c(u)\}$ and $|C_1(u)| = 1$. We can choose the unique color in $C_1(u)$ to color x to keep u 3-frugal. If $c(x) \notin \{c(y), c(z)\}$, then we obtain a linear coloring of G. Otherwise, without loss of generality, assume $c(x) = c(y)$. We can recolor v with $\alpha \notin \{c(x), c(u), c(z), c(z_1), c(z_2), c(y_1), c(y_2)\}$. It is easy to check that the obtained coloring is a linear coloring of G.

Otherwise, if $|C_2(u)| < \lfloor \frac{\Delta-1}{2} \rfloor$, then Δ is odd and $|C_1(u)| = 2$. If $\{c(y), c(z)\} \neq C_1(u)$, then we can properly color x such that v is 3-frugal and no 2-colored cycle is established to obtain a linear coloring of G. Now we assume that $\{c(y), c(z)\} = C_1(u)$. We can color x with $c(z)$. No 2-colored cycle will be induced except the cycles passing $uxvz$. If this possibility happens, then we can recolor v with $\alpha \notin \{c(x), c(u), c(y), c(z_1), c(z_2), c(y_1), c(y_2)\}$. It is easy to check that the obtained coloring is a linear coloring of G since $|\{c(u), c(v), c(x)\}| = 3$, $|\{c(y), c(z), c(v)\}| = 3$ and $|\{c(y), c(v), c(x)\}| = 3$.

Case 2. $c(u) = c(v)$ and $c(y) = c(z)$. If $|C_2(u)| = \lfloor \frac{\Delta-1}{2} \rfloor$ and Δ is odd, then we can properly color x to obtain a linear coloring of G except $c(z) = L \setminus (C_2(u) \cup \{c(u)\})$. If $c(z) = L \setminus (C_2(u) \cup \{c(u)\})$, then we erase the color on z and color x with $c(y)$. Finally, we choose $\alpha \notin \{c(v), c(x), c(z_1), c(z_2)\} \cup C_2(z_1) \cup C_2(z_2)$ to color z if $c(z_1) \neq c(z_2)$; we choose $\alpha \notin \{c(v), c(x), c(z_1)\} \cup C_2(z_1) \cup C_2(z_2) \cup (C_1(z_1) \cap C_1(z_2))$ to color z if $c(z_1) = c(z_2)$. Since $|\{c(v), c(x), c(z_1), c(z_2)\} \cup C_2(z_1) \cup C_2(z_2)| \leq \lceil \frac{\Delta}{2} \rceil$ and $|\{c(v), c(x), c(z_1)\} \cup C_2(z_1) \cup C_2(z_2) \cup (C_1(z_1) \cap C_1(z_2))| \leq \lceil \frac{\Delta}{2} \rceil$, α always exists. It is easy to verify that the obtained coloring is a linear coloring of G.