Discrete Applied Mathematics 173 (2014) 83–91

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A characterization of line graphs that are squares of graphs

Martin Milanič [a,b,*], Andrea Oversberg [c], Oliver Schaudt [d]

[a] University of Primorska, UP IAM, Muzejski trg 2, SI6000 Koper, Slovenia
[b] University of Primorska, UP FAMNIT, Glagoljaška 8, SI6000 Koper, Slovenia
[c] University of Cologne, Institute for Computer Science, Weyertal 80, 50931 Cologne, Germany
[d] Université Pierre et Marie Curie, Institut de Mathématiques de Jussieu, 4 place Jussieu, 75252 Paris, France

ARTICLE INFO

Article history:
Received 2 January 2013
Received in revised form 27 February 2014
Accepted 30 March 2014
Available online 16 April 2014

Keywords:
The square of a graph
Line graph
Linear time algorithm

ABSTRACT

The square of a graph G, denoted by G^2, is the graph obtained from G by putting an edge between two distinct vertices whenever their distance in G is at most 2. Motwani and Sudan proved that it is NP-complete to decide whether a given graph is the square of some graph. In this paper we give a characterization of line graphs that are squares of graphs, and show that if a line graph is a square, then it is a square of a bipartite graph. As a consequence, we obtain a linear time algorithm for deciding whether a given line graph is the square of some graph.

1. Introduction

Given two graphs G and H and a positive integer k, we say that G is the k-th *power* of H (and denote this by $G = H^k$) if the vertex sets of G and H coincide and two distinct vertices are adjacent in G if and only if they are at distance at most k in H. The graph H is then called a k-th *root* of G. In the case $k = 2$, we say that G is the *square* of H and H is the *square root* of G. We will say that a graph G is a *square graph* if it admits a square root.

Graph powers are basic graph transformations with a number of results about their properties in the literature [1,5,8, 11–16]. Motwani and Sudan proved in 1994 that it is NP-complete to decide whether a given graph is the square of some graph [19]. In 2006, Lau [11] proved that determining whether a given graph is the cube of some bipartite graph is NP-complete. In the same paper, he conjectured that for every fixed $k \geq 2$, recognizing k-th powers of graphs is NP-complete and that for every fixed $k \geq 3$, recognizing k-th powers of bipartite graphs is NP-complete [11]. Both conjectures were proved by Le and Nguyen [13].

It is also NP-complete to determine if a given graph has a square root that is either chordal [12], split [12], of girth four [8], or of girth five [7]. On the other hand, there are polynomial time algorithms for computing a square root that is either a tree [11,17], a bipartite graph [11], a proper interval graph [12], a block graph [14], a strongly chordal split graph [15], or a graph of girth at least six [8]. Several optimization problems are NP-complete for powers of graphs [17].

The complexity of the recognition of square graphs varies when the input graph is restricted to particular graph classes. On the one hand, the problem remains NP-complete for chordal graphs [12]. On the other hand, the problem is polynomial for planar graphs [17] and for trivially perfect graphs [18].

In this paper, we continue this line of research, focusing on line graphs. We characterize the line graphs that are the square of some graph. We show that the condition that a line graph L is the square of a graph can be stated in terms of any

* Corresponding author at: University of Primorska, UP FAMNIT, Glagoljaška 8, SI6000 Koper, Slovenia. Tel.: +386 5 611 75 84.
E-mail addresses: martin.milanic@upr.si (M. Milanič), oversberg@zpr.uni-koeln.de (A. Oversberg), schaudt@math.jussieu.fr (O. Schaudt).

http://dx.doi.org/10.1016/j.dam.2014.03.021

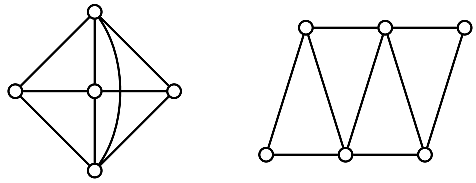

Fig. 1. The graphs K_5^- (left) and P_6^2 (right).

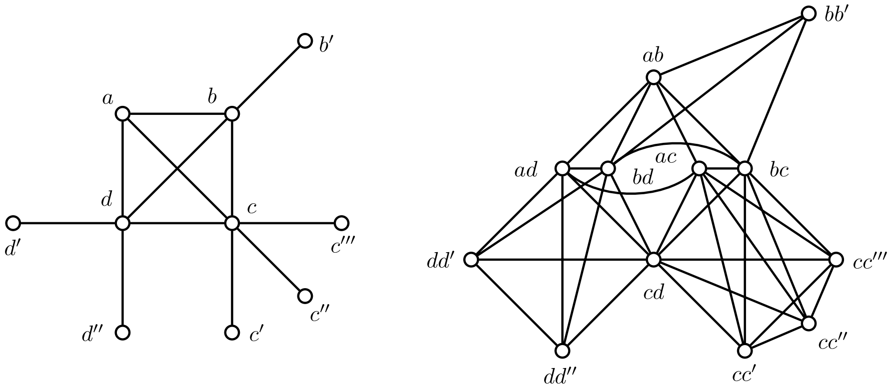

Fig. 2. A porcupine P (left) and its line graph $L(P)$ (right).

of its line roots, as follows: L is the square of a graph if and only if every connected component of each of its line roots can be obtained from a complete graph by attaching, in an arbitrary manner, pendant vertices to all but at least one vertex. (See Theorem 1 in Section 3.) Graphs obtained this way from complete graphs are called porcupines. We give a characterization of line graphs of porcupines in Theorem 2. Our characterization of line graphs that are squares of graphs also shows that line graphs with a square root have a bipartite square root, and can be used to develop a linear time algorithm for computing a (bipartite) square root (if one exists) of a given line graph.

2. Preliminaries: line graphs

All graphs in this paper will be finite, simple and undirected. Let G be a graph. The *line graph* of G, denoted by $L(G)$, has the set $E(G)$ as its vertex set and two distinct vertices $e, f \in V(L(G))$ are adjacent if and only if they share a common vertex in G. We will say that G is a *line root* of $L(G)$. Line graphs have received an enormous amount of interest in the graph theory literature. There are many papers discussing the structural and algorithmic properties of the class of line graphs. One of the classic results in this context is the characterization of line graphs in terms of nine forbidden induced subgraphs given by Beineke [4]. For our results, we need only two of these forbidden induced subgraphs. By K_n^- we denote the graph obtained from the complete graph K_n by removing an arbitrary edge. By P_n (resp. C_n) we denote the chordless path (resp. cycle) on n vertices.

Proposition 1. *A line graph does not contain K_5^- or P_6^2 as induced subgraph (see Fig. 1).*

Proof. This follows from the fact that K_5^- and P_6^2 are contained in the list of forbidden induced subgraphs given by Beineke [4]. □

3. Results

In this section, we state the main results of this paper. Proofs will be given in Section 4.

Our first result characterizes line graphs that are squares. To state this characterization, we need the notion of porcupines. A *porcupine* is a graph G obtained from a complete graph by attaching, in an arbitrary manner, pendant vertices to all but at least one vertex of the complete graph. That is, there is a partition of $V(G)$ into vertex sets U and V such that U is a non-empty clique and every member of V is a pendant vertex in G. Moreover, there is at least one vertex in U that does not have a neighbor in V. Notice that this definition of a porcupine is slightly more restrictive than the one in [9], which also allows that every vertex in U has a neighbor in V.

In particular, every complete graph is a porcupine, and the only trees that are porcupines are stars. An example of a porcupine is displayed in Fig. 2.