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Partnership formation and multinomial values[☆]

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ABSTRACT

We use multinomial values to study the effects of the partnership formation in cooperative games, comparing the joint effect on the involved players with the alternative alliance formation. The simple game case is especially considered and the application to the Catalonia Parliament (Legislature 2003–2007) is also studied.

1. Introduction

The notion of coalition of partners or *partnership* – as it will be called here – was introduced in [16]. In [3] the significance and scope of this concept were emphasized, first in cooperative games and later on for simple games, and the natural way to *impose* partnerships in a given game by means of commitments between players was also suggested. In the present paper we focus on a subfamily of probabilistic values called *multinomial (probabilistic) values*. These values were introduced in reliability by Puente [20] (see also [15]) with the name of "multibinary probabilistic values". They were independently defined by Carreras [4], for simple games only – i.e. as power indices – in a work on decisiveness where they were called "Banzhaf α-indices". Recently, Carreras and Puente [11] have given two characterizations of the multinomial values within the class of probabilistic values: one for each value and another for the whole family.

For more than a decade, our research group has been studying *semivalues*, a subfamily of probabilistic values introduced by Dubey et al. [14], characterized by anonymity, and including the Shapley value as the only *efficient* member. In the analysis of certain cooperative problems we have successfully used *binomial semivalues* [20] that include the Banzhaf value introduced by Owen [17].[1] From this experience, we feel that multinomial values (n parameters, n being the number of players) offer a deal of flexibility clearly greater than binomial semivalues (one parameter) and hence many more possibilities to introduce additional information when evaluating a game.

The aim of this paper is the application of *multinomial values* to study the effects of the partnership formation. Our first goal is to investigate how these values are modified if several players agree to form a partnership and generalize the previous

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[1] [1,6,9,10] and [12] are samples of our work in this line.

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results found by using binomial semivalues in [10]. Our second goal is the study of a real life political instance: the Catalonia Parliament during the Legislature 2003–2007.

The organization of the paper is then as follows. In Section 2, a minimum of preliminaries is provided. In Section 3, general statements for cooperative games are first given and concern the variation of the multinomial values, when a partnership is formed, and refer to (a) inner players and (b) outside players; next, a comparison is established between the multinomial values of the coalition as (i) a partnership and (ii) an alliance. In Section 4, we analyze partnerships in simple games: in this case, we determine the maximum and minimum values of the differences found for any multinomial value in the three cases mentioned above and supply games where these extreme values are attained. Proofs of the statements in Sections 3 and 4 will be found in Appendices A and B, respectively. Section 5 contains the analysis of the Catalonia Parliament if a partnership is formed. Finally, Section 6 states some conclusions.

2. Preliminaries

Let $N = \{1, 2, \dots, n\}$ denote a finite set of *players*. A *cooperative game* in N is a function $v : 2^N \to \mathbb{R}$, which assigns a real number $v(S)$ to each *coalition* $S \subseteq N$ and satisfies $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. Player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. For example, if $\emptyset \neq S \subseteq N$, the *unanimity game* u_S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. In this monotonic game, every $j \notin S$ is a null player and all members of S are symmetric players. The vector space of all games in N will be denoted as $\mathcal{G}_N$. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all v.

2.1. Probabilistic values

Following Weber's [23] axiomatic definition, $\phi : \mathcal{G}_N \to \mathbb{R}^N$ is a (group) *probabilistic value* iff it satisfies the following properties:

(i) *linearity*: $\phi[v + v'] = \phi[v] + \phi[v']$ and $\phi[\lambda v] = \lambda \phi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;
(ii) *positivity*[2]: if v is monotonic, then $\phi[v] \geq 0$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game v, then $\phi_i[v] = v(\{i\})$.

There is an interesting characterization of the probabilistic values, also in [23]: (a) given a set of $n2^{n-1}$ *weighting coefficients* $\{p_S^i : i \in N,\ S \subseteq N \backslash \{i\}\}$ such that $\sum_{S \subseteq N \backslash \{i\}} p_S^i = 1$ for each $i \in N$ and $p_S^i \geq 0$ for all $i \in N$ and $S \subseteq N \backslash \{i\}$, the expression

$$\phi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p_S^i [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N \tag{1}$$

defines a probabilistic value ϕ on N; (b) conversely, every probabilistic value can be obtained in this way; (c) the correspondence given by $\{p_S^i : i \in N,\ S \subseteq N \backslash \{i\}\} \mapsto \phi$ is one-to-one.

Thus, the payoff that a probabilistic value allocates to every player in any game is a weighted sum of his marginal contributions in the game. We quote from [23]:

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p_S^i is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game".

Among the probabilistic values, *semivalues*, introduced by Dubey et al. [14], are characterized by the *anonymity* property: $\phi_{\theta i}[\theta v] = \phi_i[v]$ for all $i \in N$, $v \in \mathcal{G}_N$ and θ, permutation on N. Alternatively, this is equivalent to saying that, if $n = |N|$, there is a vector $\{p_s\}_{s=0}^{n-1}$ such that $p_S^i = p_s$ for all $i \in N$ and all $S \subseteq N \backslash \{i\}$, where $s = |S|$, so that all coalitions of a given size share a common weight that applies to all (external) players, and hence Eq. (1) reduces to

$$\phi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

The weighting coefficients $\{p_s\}_{s=0}^{n-1}$ of any semivalue ϕ satisfy therefore two characteristic conditions:

$$\text{each } p_s \geq 0 \quad \text{and} \quad \sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1.$$

Well-known examples of semivalues are the *Shapley value* φ [21], for which $p_s = 1/n \binom{n-1}{s}$, and the *Banzhaf value* β [17], for which $p_s = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. Note that these two classical values are defined for each N.

[2] In [23] this property is called *monotonicity*, but we prefer to call to it *positivity* as in [14].