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Skeletal configurations of ribbon trees

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ABSTRACT

The straight skeleton construction creates a straight-line tree from a polygon. Motivated by moduli spaces from algebraic geometry, we consider the inverse problem of constructing a polygon whose straight skeleton is a given tree. We prove there exists only a finite set of planar embeddings of a tree appearing as straight skeletons of convex polygons. The heavy lifting of this result is performed by using an analogous version of Cauchy's arm lemma. Computational issues are also considered, uncovering ties to a much older angle bisector problem.

1. Straight skeletons

This paper is interested in the relationship between polygons and their underlying tree structures. The *medial axis* of a simple polygon P is the closure of the set of points in its interior having more than one closest point to the boundary of P. If the polygon is convex, the medial axis is a straight-line tree [8, Chapter 5], where the leaves are the vertices of the polygon, and the internal nodes are equidistant to three or more of its sides. If the polygon has a reflex vertex, however, the medial axis can have a parabolic arc.

The *straight skeleton* of a polygon is a related structure to the medial axis, which constructs a straight-line metric tree for any simple polygon. It was introduced to computational geometry by Aichholzer et al. [1], and appears in areas ranging from automated designs of roofs to origami folding problems. To construct the straight skeleton, start moving the sides of the polygon inward at equal velocity, parallel to themselves. These lines, at each point of time, bound a polygon with equal corresponding angles to the original one. Continue until the topology of the polygon traced out by this process changes. One of the two events occur:

1. *Shrink event*: When one side of the polygon shrinks to a point, two non-adjacent sides become adjacent. Continue moving all the sides inward, parallel to themselves again.
2. *Split event*: When a reflex vertex touches a side of the shrinking polygon, the polygon is split into two. Continue the inward line movement in each of them.

The straight skeleton is the straight-line tree traced out by the vertices of the shrinking polygons. Fig. 1 shows (a) the medial axis and (b) the straight skeleton of a nonconvex polygon. For convex polygons, the medial axis and the straight skeleton coincide; in this case, only shrink events occur.

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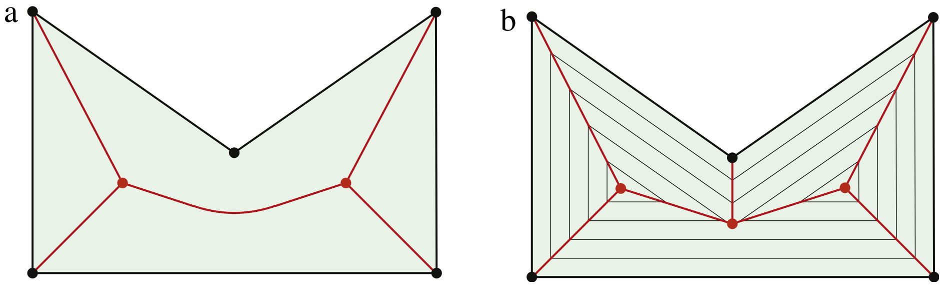

Fig. 1. (a) Medial axis and (b) straight skeleton.

This paper considers the inverse problem to this construction: Given a metric tree, construct a polygon whose straight skeleton is the tree. Section 2 provides some preliminary definitions and observations, extending the examples constructed in [2]. The notion of velocity in capturing the skeleton of a polygon is introduced in Section 3, and the main theorem is given in Section 5: For a ribbon tree T with n leaves, there exist at most $2n - 5$ configurations of T which appear as straight skeletons of convex polygons. Section 4 contains the lemma which does the heavy lifting, analogous to Cauchy's arm lemma used in the rigidity of convex polyhedra [8]. Finally, Section 6 closes with computational issues related to constructing a polygon, uncovering ties to a much older angle bisector problem.

Remark. There has been tremendous interest recently in mathematical biology, including the fields of phylogenetics and genomics. The work by Boardman [6] on the language of trees from the homotopy viewpoint has kindled numerous structures of tree spaces. The most notable could be that of Billera, Holmes, and Vogtmann [5] on a space of metric trees. Another construction involving planar trees is given in [7], where a close relationship (partly using origami foldings) is given to $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$, the real points of moduli spaces of stable genus zero algebraic curves marked with families of distinct smooth points. One can understand them as spaces of rooted metric trees with labeled leaves, resolving the singularities studied in [5] from the phylogenetic point of view.

As there exist spaces of planar metric trees, there are moduli spaces of planar polygons: Given a collection of positive real numbers $\bar{r} = \langle r_1, \ldots, r_n \rangle$, consider the moduli space of polygons in the plane with consecutive side lengths as given by $\bar{r}$, viewed as equivalence classes of planar linkages. A complex-analytic structure can be given to this space, defined by Deligne–Mostow weighted quotients [11]. From a high level, our inverse problem provides a rudimentary map between moduli spaces of metric trees and moduli spaces of polygons.

2. Preliminary properties

This section provides definitions and preliminary results, and the reader is encouraged to consult [2], where some constructive examples are provided in detail.

Definition. A *ribbon tree* is a tree (a connected graph with no cycles) for which each edge is assigned a nonnegative length, each internal vertex has degree at least three, and the edges incident to each vertex are cyclically ordered.[1] A *drawing* of a ribbon tree T is a planar straight-line embedding of T, respecting its cyclic orderings.

Definition. A polygon is *suitable* for a ribbon tree T if its straight skeleton is a drawing of T. Such a drawing is called a *skeletal configuration* of T.

Consider two natural classes of trees: stars, and more generally, caterpillars. A *star* S_n has $n + 1$ vertices, with one vertex of degree n connecting to n leaves. A *caterpillar* becomes a path if all its leaves are deleted.

Proposition 1. *There exist ribbon caterpillars without suitable polygons.*

Proof. First consider the special case of ribbon stars. Consider a ribbon star S_{3n} with edges $e_0, e_1, \ldots, e_{3n-1}$ in clockwise order. We set the edge length of e_i to be equal to x if $i \equiv 0 \bmod 3$, and equal to y otherwise. We claim that if $n \geq 3$ and the ratio x/y is sufficiently small, then this tree cannot be the straight skeleton of any polygon; see Fig. 2(a) for the case $n = 3$.

Let O be the center of the star, and A, B, C denote leaves with edges AO and BO of length y and CO of length x; see Fig. 2(b). In order to have a straight skeleton, edge BO must bisect $\angle ABC$ into two angles of measure α. Defining $\beta := \angle AOB$, note that since triangle AOB is isosceles, $\beta = \pi - 2\alpha$. As the length of x decreases relative to y, α becomes arbitrarily close to zero. Then β must approach π; specifically, β can be made greater than $2\pi/3$. Since star S_{3n} has $n \geq 3$ groups of three consecutive $\{x, y, y\}$ edges, then at least three such angles β around the center O are greater than $2\pi/3$, giving a contradiction.

[1] This is sometimes called a *fatgraph* as well [12].