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On the fractional metric dimension of graphs

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ABSTRACT

In Arumugam et al. (2013), Arumugam et al. studied the fractional metric dimension of the Cartesian product of two graphs, and proposed four open problems. In this paper, we determine the fractional metric dimension of vertex-transitive graphs, in particular, the fractional metric dimension of a vertex-transitive, distance-regular graph is expressed in terms of its intersection numbers. As an application, we calculate the fractional metric dimension of Hamming graphs and Johnson graphs, respectively. Moreover, we give an inequality for metric dimension and fractional metric dimension of an arbitrary graph, and determine all graphs for which the equality holds. Finally, we establish bounds on the fractional metric dimension of the Cartesian product of graphs. As a result, we completely solve the four open problems.

1. Introduction

Let G be a finite, simple and connected graph. We often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For any two vertices x and y of G, $d_G(x, y)$ denotes the distance between x and y, $R_G\{x, y\}$ denotes the set of vertices z such that $d_G(x, z) \neq d_G(y, z)$. If the graph G is clear from the context, $d_G(x, y)$ and $R_G\{x, y\}$ will be written $d(x, y)$ and $R\{x, y\}$, respectively. A *resolving set* of G is a subset W of $V(G)$ such that $W \cap R_G\{x, y\} \neq \emptyset$ for any two distinct vertices x and y of G. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality of all the resolving sets of G. Metric dimension was first introduced in the 1970s, independently by Harary and Melter [8] and by Slater [9]. It is a parameter that has appeared in various applications (see [3,5] for more information).

Let $f : V(G) \to [0, 1]$ be a real valued function. For $W \subseteq V(G)$, denote $f(W) = \sum_{v\in W} f(v)$. We call f a *resolving function* of G if $f(R_G\{x, y\}) \geq 1$ for any two distinct vertices x and y of G. The *fractional metric dimension*, denoted by $\dim_f(G)$, is given by

$$\dim_f(G) = \min\{|g| : g \text{ is a resolving function of } G\},$$

where $|g| = g(V(G))$. Arumugam and Mathew [1] formally introduced the fractional metric dimension of graphs and obtained some basic results. The fractional metric dimension of some product graphs was studied in [2,6,7].

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with the vertex set $V(G) \times V(H) = \{(u, v)|u \in V(G), v \in V(H)\}$, where (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$, or $v_1 = v_2$ and $\{u_1, u_2\} \in E(G)$. When there is no confusion the vertex (u, v) of $G\Box H$ will be written uv. Observe that $d_{G\Box H}(u_1v_1, u_2v_2) = d_G(u_1, u_2) + d_H(v_1, v_2)$.

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Very recently, Arumugam et al. [2] characterized all graphs G satisfying $\dim_f(G) = \frac{|V(G)|}{2}$, presented several results on the fractional metric dimension of the Cartesian product of graphs, and raised the following four open problems:

Problem 1. Determine $\dim_f(K_2 \square C_n)$ when n is odd, where K_2 is the complete graph of order 2 and C_n is a cycle of order n.

Problem 2. Determine $\dim_f(H_{n,k})$, where the Hamming graph $H_{n,k}$ is the Cartesian product of n cliques K_k.

Problem 3. Cáceres et al. [5] proved $\dim(G \square H) \geq \max\{\dim(G), \dim(H)\}$. Is a similar result true for $\dim_f(G \square H)$?

Problem 4. Let G and H be two graphs with $\dim_f(G) = \frac{|V(G)|}{2}$ and $|V(H)| \leq |V(G)|$. Is $\dim_f(H \square G) = \frac{|V(G)|}{2}$?

The motivation of this paper is to solve all these problems. In Section 2, we determine the fractional metric dimension of vertex-transitive graphs, in particular, the fractional metric dimension of a vertex-transitive, distance-regular graph is expressed in terms of its intersection numbers. As an application, we calculate the fractional metric dimension of Hamming graphs and Johnson graphs, respectively. In Section 3, we give an inequality for metric dimension and fractional metric dimension of an arbitrary graph, and determine all graphs for which the equality holds. In Section 4, we establish bounds on the fractional metric dimension of the Cartesian product of graphs.

2. Vertex-transitive graphs

For a graph G, in this paper we always assume that

$$r(G) = \min\{|R\{x, y\}| \mid x, y \in V(G), x \neq y\}. \tag{1}$$

In this section we shall express the fractional metric dimension of a vertex-transitive graph G in terms of the parameter $r(G)$, and solve Problems 1 and 2.

Lemma 2.1. *Let G be a graph with $r(G)$ as in (1). Then* $\dim_f(G) \leq \frac{|V(G)|}{r(G)}$.

Proof. Define $f : V(G) \longrightarrow [0, 1], x \longrightarrow \frac{1}{r(G)}$. For any two distinct vertices x and y, we have

$$f(R\{x, y\}) = \frac{|R\{x, y\}|}{r(G)} \geq 1,$$

which implies that f is a resolving function. Hence, $\dim_f(G) \leq |f| = \frac{|V(G)|}{r(G)}$. $\square$

A graph G is *vertex-transitive* if its full automorphism group $\mathrm{Aut}(G)$ acts transitively on the vertex set.

Theorem 2.2. *Let G be a vertex-transitive graph with $r(G)$ as in (1). Then* $\dim_f(G) = \frac{|V(G)|}{r(G)}$.

Proof. Denote $r = r(G)$. Then there exist two distinct vertices u and v such that $|R\{u, v\}| = r$. Suppose $R\{u, v\} = \{w_1, \ldots, w_r\}$. For any automorphism σ of G,

$$R\{\sigma(u), \sigma(v)\} = \{\sigma(w_1), \ldots, \sigma(w_r)\}.$$

Let f be a resolving function with $\dim_f(G) = |f|$. Then

$$f(\sigma(w_1)) + \cdots + f(\sigma(w_r)) = f(R\{\sigma(u), \sigma(v)\}) \geq 1,$$

which implies that

$$\sum_{\sigma \in \mathrm{Aut}(G)} (f(\sigma(w_1)) + \cdots + f(\sigma(w_r))) \geq |\mathrm{Aut}(G)|.$$

Since G is vertex transitive, we have

$$|\mathrm{Aut}(G)_{w_1}| \cdot |f| + \cdots + |\mathrm{Aut}(G)_{w_r}| \cdot |f| \geq |\mathrm{Aut}(G)|.$$

It follows that $\dim_f(G) = |f| \geq \frac{|V(G)|}{r}$. By Lemma 2.1 we accomplish our proof. $\square$

Arumugam et al. [2] proved that $\dim_f(K_2 \square C_n) = 2$ when n is even. Here we consider the remaining case.

Theorem 2.3. *If n is an odd integer with $n \geq 3$, then* $\dim_f(K_2 \square C_n) = \frac{2n}{n+1}$.